BASIC THEOREMS ON MULTIMAPS OF THE KKM, BROWDER, AND KAKUTANI TYPES

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Abstract. We deduce several equivalent formulations of the KKM principle, the Fan–Browder fixed point theorem, and the Kakutani fixed point theorem. Our results take the forms of a KKM type theorem, various fixed point theorems, and maximal element theorems on convex spaces in the sense of Lassonde. Especially, we give a direct transparent proof of the Kakutani theorem using our version of the KKM theorem.

1. Introduction

It is well-known that the Brouwer fixed point theorem, the Sperner lemma, the Knaster–Kuratowski–Mazurkiewicz (KKM) theorem, and a large number of results in nonlinear analysis, topology, and other fields are equivalent to each other; see [9, 14, 15]. The Kakutani fixed point theorem is one of them and very useful in many fields including mathematical economics and game theory; see [9].

In this paper, we give several results which are equivalent to the Brouwer fixed point theorem. In fact, from a KKM type theorem for convex spaces in the sense of Lassonde [7], we deduce several equivalent formulations of the Fan–Browder fixed point theorem and give a direct transparent proof of the Kakutani fixed point theorem. Our results take the forms of a KKM type theorem, various Fan–Browder type fixed point theorems, and maximal element theorems on convex spaces. Among them are generalizations of results due to Fan [4], Browder [2],

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2000 Mathematics Subject Classification. 46A55, 47H10, 54C15, 54H25.

Key Words. multimap (map), upper semicontinuous (u.s.c.), convex space, Kakutani map, fixed point, maximal element.

Kim [6], Boglin and Keiding [1], and Yannelis and Prabhakar [19]. Precisely, we are able to give new proofs of those known results in much more general forms.

2. On the KKM maps

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^{-}(y) = \{x \in X : y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) : x \in A\}$.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X.

For a set D, let $\langle D \rangle$ denote the set of nonempty finite subsets of D.

Let X be a subset in a vector space and D a nonempty subset of X. Then (X, D) is called a *convex space* if convex hulls of any $N \in \langle D \rangle$ are contained in X and X has a topology that induces the Euclidean topology on such convex hulls; see Park [8]. If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [7]. In a convex space (X, D), a subset $A \subset X$ is said to be *convex* (or *D-convex*) if for each $N \in \langle D \rangle$, $N \subset A$ implies $\operatorname{co} N \subset A$. We also define

$$\operatorname{co} A := \bigcap\{B : B \text{ is a convex subset of } (X, D) \text{ containing } A\}$$

for each $A \subset X$.

The following is well-known:

The KKM Principle. *Let D be the set of vertices of an n-simplex Δ_n and $F : D \multimap \Delta_n$ be a KKM map (that is, $\operatorname{co} N \subset F(N)$ for each $N \in \langle D \rangle$) with open [resp. closed] values. Then $\bigcap_{v \in D} F(v) \neq \emptyset$.*

The "closed" version is the celebrated Knaster–Kuratowski–Mazurkiewicz theorem in 1929. The "open" version is due to Kim [6] and Shih and Tan [18] in 1987. The two versions are equivalent; see [15].

As in the previous works [4, 11, 12], from the KKM principle, we can obtain the following version of the KKM theorem for convex spaces:

Theorem 1. *Let* (X, D) *be a convex space and* $F : D \multimap X$ *a muiltimap such that*

(1) $F(z)$ *is open* [*resp. closed*] *for each* $z \in D$; *and*

(2) F *is a KKM map.*

Then $\{F(z)\}_{z\in D}$ *has the finite intersection property* (*that is, for any* $N \in \langle D\rangle$, *we have* $\bigcap_{z\in N} F(z) \neq \emptyset$.)

Note that the KKM principle and Theorem 1 are equivalent.

3. On the Browder maps

In 1968, F. Browder [2] obtained the following fixed point theorem:

Theorem (Browder). *Let* K *be a nonempty compact convex subset of a Hausdorff topological vector space. Let* T *be a map of* K *into* 2^K, *where for each* $x \in K$, $T(x)$ *is a nonempty convex* [*resp. open*] *subset of* K. *Suppose further that for each* $y \in K$, $T^-(y)$ *is open* [*resp. nonempty and convex*] *in* K. *Then there exists* $x_0 \in K$ *such that* $x_0 \in T(x_0)$.

Note that Browder's result is a reformulation of Fan's geometric lemma [4] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it has been called the Fan–Browder fixed point theorem. Later the Hausdorffness of X is known to be superfluous, and the Fan–Browder theorem is known to be equivalent to the Brouwer fixed point theorem.

Browder [2] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. For further developments on generalizations and applications of the Fan–Browder theorem, we refer to Park [9, 10, 12, 14]. Usually, a multimap with nonempty convex values and open fibers is called a *Browder map*.

In this section, we deduce various generalizations of the Fan–Browder theorem in the frame of our convex spaces.

From Theorem 1, we obtain our main result in this section as follows:

Theorem 2. *Let* (X, D) *be a convex space and* $S : D \multimap X$, $T : X \multimap X$ *multimaps. Suppose that*

(1) $S(z)$ *is open* [*resp. closed*] *for each* $z \in D$;

(2) $\operatorname{co} S^-(y) \subset T^-(y)$ *for each* $y \in X$; *and*

(3) $X = S(M)$ *for some* $M \in \langle D \rangle$.

Then T *has a fixed point* $x_0 \in X$; *that is,* $x_0 \in T(x_0)$.

Proof. Define a map $F : D \multimap X$ by $F(z) := X \backslash S(z)$ for each $z \in D$. Then each $F(z)$ is closed [resp. open] by (1), and

$$\bigcap_{z \in M} F(z) = X \backslash \bigcup_{z \in M} S(z) = X \backslash X = \emptyset$$

by (3). Therefore, the family $\{F(z)\}_{z \in D}$ does not have the finite intersection property, and hence, F is not a KKM map by Theorem 1. Thus, there exists an $N \in \langle D \rangle$ such that $\operatorname{co} N \not\subset F(N) = \bigcup\{X \backslash S(z) : z \in N\}$. Hence, there exists an $x_0 \in \operatorname{co} N$ such that $x_0 \in S(z)$ for all $z \in N$; that is, $N \subset S^-(x_0)$. Therefore, $x_0 \in \operatorname{co} N \subset \operatorname{co} S^-(x_0) \subset T^-(x_0)$ by (2). This implies $x_0 \in T(x_0)$ and completes our proof.

Proof of Theorem 1 using Theorem 2. Choose any $N \in \langle D \rangle$. Let $S : N \multimap X$ and $T : X \multimap X$ be multimaps defined by $S(z) := X \backslash F(z)$ for $z \in N$ and $T^-(y) := \operatorname{co} S^-(y)$ for $y \in X$. Then S is closed-valued [resp. open-valued] and T^- is convex-valued. Suppose that $\bigcap_{z \in N} F(z) = \emptyset$, contrary to the conclusion. Then

$$X = X \backslash \bigcap_{z \in N} F(z) = \bigcup_{z \in N} (X \backslash F(z)) = \bigcup_{z \in N} S(z) = S(N).$$

Therefore, by Theorem 2, T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$ or $x_0 \in T^-(x_0) = \operatorname{co} S^-(x_0)$. Hence, there exist $z_1, z_2, \cdots, z_n \in S^-(x_0) \subset N$ such that $x_0 \in \operatorname{co}\{z_i\}_{i=1}^n$. Since $x_0 \in S(z_i) = X \backslash F(z_i)$, we have $x_0 \notin F(z_i)$ for all $i = 1, 2, \cdots, n$. Therefore, we have an $x_0 \in \operatorname{co}\{z_i\}_{i=1}^n$ such that $x_0 \notin \bigcup_{i=1}^n F(z_i)$. Hence F is not a KKM map. This is a contradiction.

Corollary 2.1. *Let* (X, D) *be a convex space and* $P : X \multimap X$. *If there exist* $z_1, z_2, \cdots, z_n \in D$ *and nonempty open* [*resp. closed*] *subsets* $G_i \subset P^-(z_i)$ *for*

$i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^n G_i$, then the map $\mathrm{co}\, P : X \multimap X$ has a fixed point.

Proof. Let $M := \{z_1, z_2, \cdots, z_n\} \subset D$ and define a map $S : D \multimap X$ by $S(z_i) := G_i$ for each $z_i \in M$ and $S(z) := \emptyset$ for each $z \in D \backslash M$. Then (1) $S(z)$ is open [resp. closed] for each $z \in D$; and (3) $X = S(M)$.

Since $S(z_i) = G_i \subset P^-(z_i)$ for all $z_i \in M$ and $\emptyset = S(z) \subset P^-(z)$ for all $z \in D \backslash M$, we have $S(z) \subset P^-(z)$ for all $z \in D$. Therefore, for any $y \in X$ and $z \in D$, $y \in S(z)$ implies $y \in P^-(z)$, and hence, $z \in S^-(y)$ implies $z \in P(y)$. Then we have $S^-(y) \subset P(y)$ for each $y \in X$.

Let us define a map $T : X \multimap X$ by $T^-(y) := \mathrm{co}\, P(y)$ for $y \in X$. Then we have (2) $\mathrm{co}\, S^-(y) \subset \mathrm{co}\, P(y) = T^-(y)$ for each $y \in X$. Therefore, all of the requirements of Theorem 2 are satisfied. Hence, T has a fixed point $x_0 \in T(x_0)$ and hence $x_0 \in T^-(x_0) = \mathrm{co}\, P(x_0)$. This completes our proof.

The open case of Corollary 2.1 is equivalent to the following:

Corollary 2.2. *Let (X, D) be a convex space and $T : X \multimap X$ a multimap such that*

(1) $T(x)$ is convex for each $x \in X$; and

(2) $X = \bigcup_{i=1}^n \mathrm{Int}\, T^-(z_i)$ for some $\{z_1, z_2, \cdots, z_n\} \subset D$.

Then T has a fixed point.

Proof. In Corollary 2.1, put $G_i := \mathrm{Int}\, T^-(z_i)$. Then G_i are open, $G_i \subset T^-(z_i)$ for each i, and $X = \bigcup_{i=1}^n G_i$. Hence, by Corollary 2.1, $T = \mathrm{co}\, T$ has a fixed point.

Proof of the open case of Corollary 2.1 using Corollary 2.2. Let $T := \mathrm{co}\, P : X \multimap X$. Then $T(x)$ is convex for each $x \in X$ and $G_i \subset P^-(z_i) \subset (\mathrm{co}\, P)^-(z_i) = T^-(z_i)$ for each i. Since each G_i is open in X, we have $G_i \subset \mathrm{Int}\, T^-(z_i) \subset X$ for each i. Therefore, $X = \bigcup_{i=1}^n G_i = \bigcup_{i=1}^n \mathrm{Int}\, T^-(z_i)$. Hence, by Corollary 2.2, $T = \mathrm{co}\, P$ has a fixed point.

It is easy to see that Corollary 2.2 implies the Fan–Browder fixed point theorem, and not conversely.

From Theorem 2, we immediately have the following equivalent formulation:

Theorem 3. *Let (X, D) be a convex space, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that*

(1) *$S^-(z)$ is open [resp. closed] for each $z \in D$;*
(2) *$\operatorname{co} S(x) \subset T(x)$ for each $x \in X$; and*
(3) *$X = S^-(M)$ for some $M \in \langle D \rangle$.*

Then T has a fixed point $x_0 \in X$.

Corollary 3.1. *Let (X, D) be a convex space and $P : X \multimap D$ a map such that*

(1) *$P^-(z)$ is open [resp. closed] for each $z \in D$; and*
(2) *$x \notin \operatorname{co} P(x)$ for all $x \in X$.*

Then X can not be covered by a finite number of $P^-(z)$'s for $z \in D$.

Proof. Suppose $X = P^-(M)$ for some $M \in \langle D \rangle$. By putting $S := P$ and $T := \operatorname{co} P : X \multimap X$, all of the requirements of Theorem 3 are satisfied. Hence $T = \operatorname{co} P$ has a fixed point. This contradicts (2).

Corollary 3.2. *Let (X, D) be a convex space and $P : X \multimap D$ a map such that*

(1) *$P^-(z)$ is open for each $z \in D$;*
(2) *$x \notin \operatorname{co} P(x)$ for all $x \in X$; and*
(3) *X is compact.*

Then there exists $x^* \in X$ such that $P(x^*) = \emptyset$.

Proof. Suppose otherwise, i.e., $P(x) = \emptyset$ for all $x \in X$. Then, for each $x \in X$, there exists a $z \in D$ such that $z \in P(x)$ or $x \in P^-(z)$. Hence, X is covered by $P^-(z)$'s for $z \in D$. Since each $P^-(z)$ is open and X is compact, X has a finite subcover by $P^-(z)$'s. This contradicts Corollary 3.1.

Any binary relation R in a set X can be regarded as a map $T : X \multimap X$ (may have empty values) and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a map $T : X \multimap X$ if $T(x_0) = \emptyset$.

The Fan–Browder fixed point theorem is used by Borglin and Keiding [1] and Yannelis and Prabhakar [19] to obtain existence of maximal elements in mathematical economics as follows:

Corollary 3.3. *Let X be a compact convex space and $T : X \multimap X$ a map such that*

(1) $T^{-}(y)$ *is open for each* $y \in X$; *and*

(2) *for each* $x \in X$, $x \notin \operatorname{co} T(x)$.

Then T has a maximal element.

Note that Corollary 3.3 follows from Corollary 3.2 with $X = D$ and that the Fan–Browder theorem is equivalent to Corollary 3.3.

It is well-known that the KKM principle and the Browder theorem are equivalent to the Brouwer fixed point theorem; see [9]. Therefore, we have

Conclusion 1. The Brouwer fixed point theorem is equivalent to any of Theorems 1-3 and Corollaries 2.1, 2.2, 3.1 - 3.3.

4. On the Kakutani maps

For a topological space X and a convex space Y, we define

$T \in \mathbb{K}(X, Y) \iff T : X \multimap Y$ is a *Kakutani map*; that is, T is u.s.c. with nonempty compact convex values.

Let Δ be any convex hull of a nonempty finite subset of a convex space. Then we have the following [5] from Theorem 1:

Theorem 4 (Kakutani). *Any $T \in \mathbb{K}(\Delta, \Delta)$ has a fixed point $x_0 \in \Delta$; that is, $x_0 \in T(x_0)$.*

Proof. Let Δ be equipped with the Euclidean metric $d(x, y) = ||x - y||$ for $x, y \in \Delta$. Let $\epsilon > 0$. Since Δ is compact, there exists $D := \{x_1, x_2, \cdots, x_n\} \in \langle \Delta \rangle$ such that $\Delta \subset \bigcup_{i=1}^{n} B(x_i, \epsilon/2)$, where B denotes the open ball. Then (Δ, D) becomes a convex space.

Define a multimap $F : D \multimap \Delta$ by

$$F(x_i) := \{x \in \Delta : T(x) \cap \overline{B}(x_i, \epsilon/2) = \emptyset\}.$$

for each i, where $\overline{B}$ denotes the closed ball. Since T is u.s.c., each $F(x_i)$ is open in Δ. Moreover, we have

$$\bigcap_{i=1}^{n} F(x_i) = \{x \in \Delta : T(x) \cap \bigcup_{i=1}^{n} \overline{B}(x_i, \epsilon/2) = \emptyset\} = \emptyset.$$

Therefore, by Theorem 1, F can not be a KKM map for the convex space (Δ, D); that is, there exist $\{x_{i_1}, \cdots, x_{i_k}\} \subset D$ and $x_\epsilon \in \text{co}\{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$ such that $x_\epsilon \notin \bigcup_{j=i}^{k} F(x_{i_j})$. Hence, $T(x_\epsilon) \cap \overline{B}(x_{i_j}, \epsilon/2) \neq \emptyset$ for all $j = 1, 2, \cdots, k$. Let

$$M := \{y \in \Delta : T(x_\epsilon) \cap B(y, \epsilon) \neq \emptyset\}.$$

Then, using the properties of the norm $||\cdot||$, it is routine to check that M is convex from the fact that Δ, $T(x_\epsilon)$, and $B(y, \epsilon)$ are all convex. Since $x_{i_j} \in M$ for all j, we have $x_\epsilon \in M$ and hence $T(x_\epsilon) \cap B(x_\epsilon, \epsilon) \neq \emptyset$. Therefore, we have a $y_\epsilon \in T(x_\epsilon)$ such that $||x_\epsilon - y_\epsilon|| < \epsilon$. Since Δ is compact, the net $\{y_\epsilon\}$ in Δ has a convergent subnet with a limit $x_0 \in \Delta$. Then the corresponding subnet of $\{x_\epsilon\}$ also converges to x_0. Since T is u.s.c. with closed values, the graph of T is closed and hence we have $x_0 \in T(x_0)$. This completes our proof.

In [5], Kakutani applied his theorem to give simple proofs of the von Neumann minimax theorem in 1928 and von Neumann's intersection lemma in 1937. Later, Kakutani's theorem was made by Nash in 1950 in his proof of the existence of an equilibrium for a finite game. It was followed by several hundred applications in the theory of games, economic theory, mathematical programming, control theory, and theory of differential equations; see [9].

Note that the Kakutani theorem 4 implies the open version of the KKM principle; see [6].

By following the proof of Theorem 4, we can obtain a variation of Theorem 4 as follows:

Theorem 4′. *Let Δ be a compact convex subset of a metric t.v.s. E whose balls are convex. Then any $T \in \mathbb{K}(\Delta, \Delta)$ has a fixed point.*

Note that Theorem 4′ includes historically well-known fixed point theorems due to Brouwer, Schauder, Kakutani, Bohnenblust and Karlin, and Rassias; for the literature, see [9]. We already deduced much more general results than Theorem $4'$ using the KKM theorem 1; see [16,17].

We need the following due to S.Y. Chang [3]:

Lemma [3]. *Let X be a topological space, Y a convex space, and $A : X \multimap Y$. If there exist nonempty closed subsets $F_i \subset A^-(y_i)$ for $i = 1, 2, \cdots, n$ such that $\bigcup_{i=1}^n F_i = X$, then there exists a $B \in \mathbb{K}(X, \mathrm{co}\{y_1, y_2, \cdots, y_n\})$ such that $B(x) \subset \mathrm{co}\, A(x)$ for all $x \in X$.*

From the Kakutani theorem with the aid of Lemma, we obtain the following particular form of the closed version of Corollary 2.1.

Corollary 4.1. *Let X be a convex space and $A : X \multimap X$. If there exists $y_1, y_2, \cdots, y_n \in X$ and nonempty closed subsets $F_i \subset A^-(y_i)$ for $i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^n F_i$, then the map $\mathrm{co}\, A : X \multimap X$ has a fixed point $x_0 \in X$, that is, $x_0 \in \mathrm{co}\, A(x_0)$.*

Proof. By Lemma [3] with $X = Y$, there exists a map $B|_\Delta \in \mathbb{K}(\Delta, \Delta)$ where $\Delta = \mathrm{co}\{y_1, y_2, \cdots, y_n\}$. By Theorem 4, $B|_\Delta$ has a fixed point $x_0 \in \Delta$; that is, $x_0 \in B(x_0) \subset \mathrm{co}\, A(x_0)$.

From Corollary 4.1, we can deduce the closed version of Theorem 2 as follows:

Proof of Theorem 2 using Corollary 4.1. Let $N =: \{y_1, y_2, \cdots, y_n\}$, $F_i := S(y_i)$, and $A : X \multimap X$ be defined by $A(x) := T^-(x)$ for $x \in X$. Then $A^-(y_i) = T(y_i) \supset S(y_i) = F_i$ since $S^-(y) \subset T^-(y)$ for each $y \in X$ by (2). Moreover, $X = \bigcup_{i=1}^n F_i$ by (3). Therefore, by Corollary 4.1, there exists a point $x_0 \in X$ such that $x_0 \in \mathrm{co}\, A(x_0) = \mathrm{co}\, T^-(x_0) = T^-(x_0)$ by (2) and hence $x_0 \in T(x_0)$. This completes our proof.

We already noted that Theorem 2 is eqivalent to the KKM principle, and hence to the Brouwer fixed point theorem. Therefore, we have

Conclusion 2. *The Brouwer fixed point theorem is equivalent to any of Theorems 1-4 and Corollaries 2.1, 2.2, 3.1-3.3 and 4.1.*

Finally, it should be noticed that most of results in this paper can be generalized in the framework of generalized convex spaces or G-convex spaces due to the author [10, 11, 12].

Acknowledgement. A part of this paper is given as a lecture delivered at the 7th ICNFAA, Kyungnam University, Masan, and Gyeongsang National University, Chinju, Korea, August 5-10, 2001. The author would like to express his gratitude to the organizers.

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