ALMOST FIXED POINT THEOREMS OF THE FORT TYPE

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Abstract. From the KKM principle, we deduce an almost fixed point theorem generalizing results of Fort and Smart. from this theorem, we obtain fixed point theorems of Brouwer, Schauder, Tychonoff, hukuhara, and Rassias.

In 1954, M. K. Fort, Jr. [F] showed that any open disk in the Euclidean plane has the almost fixed point property. This was extendeed to any normed space by D. R. Smart [S]in 1993.

Our aim in this paper is to show that those results can be generalized to convex subsets of locally convex Hausdorff topological vector spaces or of metric topological vector spaces whose open balls are convex.

Our arguments are based on the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem [K]. For other result applications of the KKM theorem can be seen in the author [ ].

KKM principle. *Let D be the set of vertices of a simplex S and $F : D \multimap S$ a multimap with closed values such that*

$$\mathbf{co}\, N \subset F(N) \text{ for each } N \subset D.$$

Then $\bigcap_{z \in D} F(z) \neq \emptyset$.

It is also well-known that the following easily follows from the KKM principle; see Fan [F].

Lemma. *Let X be a subset of a topological vector space, D a nonempty subset of X such that $\mathbf{co}\, D \subset X$, and $F : D \multimap X$ a KKM map with closed values. Then $\{F(z)\}_{z \in D}$ has the finite intersection property,*

Note that a map $F : D \multimap X$ is called a *KKM map* if

$$\mathbf{co}\, N \subset F(N) \text{ for each finite subset } N \text{ of } D.$$

From now, tv.s. means a Hausdorff topological vector space.

From Lemma, we have our main result as follows:

Theorem 1. *Let X be a nonemprty convex subset of a locally convex t,v.s. E and $f : X \to \overline{X}$ a continuous map such that $\overline{f(X)}$ is totally bounded. Then f has the almost fixed point property; that is for any eighborhood U of 0 in E, there exists a point $x_U \in X$ such that $f(x_U) \in x_U + U$.*

Proof. For any neighborhood U of 0 in E, we choose a convex open neighborhood V on 0 in E such that $V+V \subset U$. Since $\overline{f(X)}$ is totally boubded in $\overline{X}$, there is an $D' \in \langle\overline{f(X)}\rangle$ such that $\overline{f(X)} \subset D'+V$. Let $D' := \{y_1, y_2, \cdots, y_n\} \subset \overline{X}$. Choose a subset $D : \{x_1, x_2, \cdots, x_n\} \subset X$ such that $y_i - x_i \in V$ for each $i = 1, 2, \cdots, n$. Let $F : D \to \overline{X}$ a multimap defined by

$$F(x_i) = \overline{f(X)} \backslash (y_i + V) \text{ for each } x_i \in D.$$

Then F is closed-valued and

$$\bigcap_{i=1}^{n} F(x_i) = \overline{f(X)} \backslash \bigcup_{i=1}^{n} (y_i + V) = \emptyset.$$

Since $f : X \to \overline{X}$ is continuous and

$$\bigcap_{i=1}^{n} f^{-1} \circ F(x_i) = f^{-1}(\bigcap_{i=1}^{n} F(x_i)) = \emptyset,$$

$f^{-1} \circ F : D \multimap X$ is closed-valued and not a KKM map. Therefore, by Lemma, there exists an $A \in \langle D\rangle$ such that $\mathbf{co}\, A \not\subseteq f^{-1} \circ F(A)$. Let $A = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$. Then there exists an $x_U \in \mathbf{co}\, A$ and $f(x_U) \notin F(x_{I_j}$ for all $j = 1, 2, \cdots, k$. Hence $f(X_U) \in y_{i_j} + V$. Let $x_U = \sum_{j=1}^{k} \lambda_{i_j} x_{i_j}$ where $\lambda_{i_j} \in [0, 1]$ and $\sum_{i=1}^{k} \lambda_{i_j} = 1$. Since $f(x_U) \in x_{i_j} + (y_{i_j} - x_{i_j}) + V \subset x_{i_j} + V + V$ and V is convex, we have

$$f(x_U) = \sum_{j=1}^{k} \lambda_{i_j} f(x_U) \in \sum_{j=1}^{k} \lambda_{i_j} x_{i_j} + V + V = x_U + V + V \subset x_U + U.$$

This completes our proof.

Corollary 1. *Let X be a nonempty convex subset of a locally convex t.v.s. E and $f : X \to X$ a continuous map. If X is totally bounded, then f has the almost fixed point property.*

Proof. Note that $f : X \to X$ can be regarded as $f : X \to \overline{X}$. Since $x(X) \subset X$ is totally bounded, so is $\overline{f(X)}$. Now, the conclusion follows from Theorem 1.

Example. (1) If E is a metric t.v.s. whose balls are convex, then Theorem 1 and corollary 2 hold.

Note that Smart [S] obtained Theorem 1 and Corollary for the following cases:

(2) E is a normed vector space.

(3)X is an open ball in a normed vector space.

(4) X is an open ball in $\mathbb{R}^n$.

More early, Fort [F] obtained Corollary 2 when

(5) X is an open disk in $\mathbb{R}^2$.

Theorem 1 or Corollary 1 does not guarantee any existence of fixed points of f.

Example. Let $X = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disk in $\mathbb{R}^2$ and $f : X \to \overline{X}$ a continuous map such that $f(x, y) = (x, \sqrt{1 - x^2})$ for all $(x, y) \in X$.

However, we have the following from Theorem 1:

Theorem 2. *Let X be a nonempty convex subset of a locally convex t.v.s. E and $f : X \to X$ a continuous map which is compact (that is, $f(X)$ is contained in a compact subset of X). Then f has a fixed point $x_0 \in X$, that is, $x_0 = f(x_0)$.*

Proof. Note that f is closed (that is, has the closed grapy) and that any closed map having the almost fixed point property has a fixed point. Therefore Theorem 2 follows from Theorem 1.

Theorem 2 was obtained by Hukuhara [H] with different proof, and includes fixed point theorem duo to Brouwer (for an n-simplex X), Schauder (for a normed vector space E), and Tychonoff (for a compact convex subset X).

We have one more

Corolary 2. *Let X be a nonempty convex subset of a metric t.v.s. E whose balls are convex and $f : X \to X$ a compact continuous map. Then f has a fixed point.*

If X itself is compact, then Corollary 2 reduces to a result of Rassias [R].

Finally, note that, since the KKM principle is equivalent to the Brouwer fixed point theorem, each of Theorems 1, 2 and corollary 1, 2 is also equivalent to the Brouwer theorem.

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