COINCIDENCE, ALMOST FIXED POINT, AND MINIMAX THEOREMS ON GENERALIZED CONVEX SPACES

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Abstract. From the KKM principle for G-convex spaces, we deduce some coincidence theorems for multimaps having the KKM property. These coincidence theorems are applied to obtain Fan–Browder type theorems for G-convex spaces, almost fixed point theorems for multimaps defined on convex subsets of t.v.s., a fixed point theorem for the better admissible class of multimaps, minimax theorems, and extension theorems for monotone sets. Moreover, the KKM principle is applied to almost fixed point results for u.s.c. or l.s.c. multimaps on LG-spaces.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or others of the corresponding problem under consideration. The first result on the nonempty intersection was the celebrated Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle) in 1929, which is concerned with certain types of multimaps called the KKM maps.

The KKM theory, first called by the author [12,13], is the study of KKM maps and their applications. Nowadays, it would be better to regard as the study of

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applications of various equivalent formulations of the KKM principle. At the begining, the theory was mainly devoted to study on convex subsets of topological vector spaces (t.v.s.). Later, it has been extended to convex spaces by Lassonde, and to C-spaces (or H- spaces) by Horvath and others. Recently the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author; for details, see [19-23,25] and references therein.

In the KKM theory, there have appeared a number of coincidence theorems with many significant applications. In this paper, from the KKM principle for G-convex spaces, we deduce some coincidence theorems related to certain broad classes of multimaps (having the KKM property). These coincidence theorems are applied to obtain Fan–Browder type theorems for G-convex spaces, almost fixed point theorems for multimaps defined on convex subsets of t.v.s., a fixed point theorem for the better admissible class of multimaps, minimax theorems, and extension theorems for monotone sets. Moreover, the KKM principle is applied to almost fixed point results for u.s.c. or l.s.c. multimaps on LG-spaces.

2. The KKM theorem for G-convex spaces

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^{-}(y) = \{x \in X \,|\, y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \,|\, x \in A\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) [resp. *lower semicontinuous* (l.s.c.)] if for each closed [resp. open] set $B \subset Y$, $F^{-}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$ is closed [resp. open] in X.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ having $n+1$ elements, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J := \Gamma(J)$.

Note that Δ_n is an n-simplex with vertices $v_0, v_1, \cdots, v_n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$.

Examples of G-convex spaces can be found in [20,21,23] and references therein.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if

$$\Gamma_N \subset F(N) \quad \text{for each } N \in \langle D \rangle.$$

The following is a KKM theorem for G-convex spaces [20,21]:

Theorem 2.1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*

(1) *F has closed [resp. open] values; and*

(2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

Remark. There have appeared several variations of Theorem 2.1; see [21,25].

3. Coincidence theorems

Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. A multimap $F : X \multimap Y$ is said to have *the KKM property* if, for any map $G : D \multimap Y$ with closed [open] values satisfying

$$F(\Gamma_A) \subset G(A) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

the family $\{G(z)\}_{z \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(X, Y) := \{F : X \multimap Y \mid F \text{ has the KKM property}\}.$$

Some authors use the notation $KKM(X, Y)$. Note that $1_X \in \mathfrak{K}(X, X)$ by Theorem 2.1. Moreover, if $F : X \to Y$ is a continuous single-valued map or if $F : X \multimap Y$ has a continuous selection, then it is easy to check that $F \in \mathfrak{K}(X, Y)$. Note that there are many known selection theorems due to Michael and others.

From now on, $\mathfrak{KC}$ denote the class $\mathfrak{K}$ for closed-valued maps G, and $\mathfrak{KO}$ for open-valued maps G.

From Theorem 2.1, we derived the following basic coincidence theorem in [22]:

Theorem 3.1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a topological space, $S : D \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{KC}(X, Y)$. Suppose that*

(1) *S has open values*;
(2) *for each $y \in F(X)$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$*; *and*
(3) *$\overline{F(X)} \subset S(N)$ for some $N \in \langle D \rangle$.*

Then F and T have a coincidence point $x_* \in X$; that is, $F(x_*) \cap T(x_*) \neq \emptyset$.

Theorem 3.1 is applied in [22] to the Fan–Browder theorem, ω-connected spaces, and Φ-spaces.

Similarly, for the class $\mathfrak{KO}(X, Y)$, we have the following basic coincidence theorem:

Theorem 3.1′. *Let* $(X, D; \Gamma)$ *be a* G*-convex space,* Y *a topological space,* $S : D \multimap Y$*,* $T : X \multimap Y$*, and* $F \in \mathfrak{K}\mathfrak{O}(X, Y)$*. Suppose that*

(1) S *has closed values*;

(2) *for each* $y \in F(X)$*,* $M \in \langle S^-(y) \rangle$ *implies* $\Gamma_M \subset T^-(y)$*; and*

(3) $Y = S(N)$ *for some* $N \in \langle D \rangle$*.*

Then F *and* T *have a coincidence point* $x_* \in X$*; that is,* $F(x_*) \cap T(x_*) \neq \emptyset$*.*

Proof. Define a map $G : D \multimap Y$ by $G(z) := Y \backslash S(z)$ for each $z \in D$. Then G has open values and

$$\bigcap_{z \in N} G(z) = \bigcap_{z \in N} (Y \backslash S(z)) = Y \backslash \bigcup_{z \in N} S(z) \subset Y \backslash Y = \emptyset.$$

Therefore $\{G(z)\}_{z \in D}$ does not have the finite intersection property.

Since $F \in \mathfrak{K}\mathfrak{O}(X, Y)$, we have

$$F(\Gamma_A) \not\subset G(A) \qquad \text{for some} \quad A \in \langle D \rangle.$$

Hence, there exists a $y_0 \in F(\Gamma_A) \subset Y$ such that $y_0 \notin G(z) = Y \backslash S(z)$ for all $z \in A$. Therefore, $y_0 \in S(z)$ or $z \in S^-(y_0)$ for all $z \in A$ and hence $A \in \langle S^-(y_0) \rangle$. Since $y_0 \in F(X)$, by (2), we have $\Gamma_A \subset T^-(y_0)$ and hence $y_0 \in F(\Gamma_A) \subset F(T^-(y_0))$. Therefore, there exists an $x_* \in T^-(y_0)$ such that $y_0 \in F(x_*)$, and hence we have $y_0 \in F(x_*) \cap T(x_*)$. This completes our proof.

Remark. It would be possible to replace the class $\mathfrak{K}$ in this paper by the so-called S-KKM class introduced by some authors. However, we will not do this in the present paper.

4. The Fan–Browder fixed point theorem and maximal elements

By putting $X = Y$ and $F = 1_X$ in Theorems 3.1 and 3.1′, we have a general form of the Fan–Browder theorem for G-convex spaces:

Theorem 4.1. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ two maps satisfying*

(1) *for each $z \in D$, $S(z)$ is open [resp. closed];*

(2) *for each $y \in X$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$; and*

(3) *$X = S(N)$ for some $N \in \langle D \rangle$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

Theorem 4.1 is obtained in [22] and applied to various forms of the Fan–Browder theorem, the Ky Fan intersection theorem, and the Nash equilibrium theorem for G-convex spaces.

From Theorem 4.1, we deduce the following new result:

Theorem 4.2. *Let $(X \supset D; \Gamma)$ be a G-convex space and $A : X \multimap X$ be a multimap such that $A(x)$ is Γ-convex for each $x \in X$. If there exist $z_1, z_2, \cdots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^n G_i$, then A has a fixed point.*

Proof. Let $N := \{z_1, z_2, \cdots, z_n\} \subset D$ and define a map $S : D \multimap X$ by $S(z_i) := G_i$ for each $z_i \in N$ and $S(z) := \emptyset$ for each $z \in D \backslash N$. Then (1) $S(z)$ is open [resp. closed] for each $z \in D$; and (3) $X = S(N)$.

Since $S(z_i) := G_i \subset A^-(z_i)$ for all $z_i \in N$ and $\emptyset = S(z) \subset A^-(z)$ for all $z \in D \backslash N$, we have $S(z) \subset A^-(z)$ for all $z \in D$. Therefore, for any $y \in X$ and $z \in D$, $y \in S(z)$ implies $y \in A^-(z)$, and hence, $z \in S^-(y)$ implies $z \in A(y)$. This shows $S^-(y) \subset A(y)$ for each $y \in X$.

Let us define a map $T : X \multimap X$ by $T^-(y) := A(y)$ for $y \in X$. Then we have $S^-(y) \subset A(y) = T^-(y)$ and $T^-(y)$ is Γ-convex for each $y \in X$. Hence, condition (2) of Theorem 4.1 holds. Therefore, by Theorem 4.1, T has a fixed point $x_0 \in X$, and hence, $x_0 \in T^-(x_0) = A(x_0)$. This completes our proof.

Any binary relation R in a set X can be regarded as a multimap $T : X \multimap X$ (may have empty values) and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a multimap $T : X \multimap X$ if $T(x_0) = \emptyset$. Similarly, a point $y_0 \in X$ is called a *minimal element* of $T : X \multimap X$ if $T^-(y_0) = \emptyset$.

The Fan–Browder type fixed point theorem for topological vector spaces is used by Borglin and Keiding [2] and Yannelis and Prabhakar [27] to the existence of maximal elements in mathematical economics. We give a generalization of their result as follows:

Theorem 4.3. *Let $(X, D; \Gamma)$ be a compact G-convex space and $S : X \multimap D$, $T : X \multimap X$ two maps such that*

(1) *$S^-(z)$ is open for each $z \in D$;*

(2) *for each $x \in X$, $N \in \langle S(x) \rangle$ implies $\Gamma_N \subset T(x)$; and*

(3) *for each $x \in X$, $x \notin T(x)$.*

Then there exists an $\widehat{x} \in X$ such that $S(\widehat{x}) = \emptyset$.

Proof. Suppose $S(x) \neq \emptyset$ for all $x \in X$. Then X is covered by $\{S^-(z) : z \in D\}$. Since X is compact, we have $X = S^-(N)$ for some $N \in \langle D \rangle$. Now, by applying Theorem 4.1 with (S^-, T^-) instead of (S, T), we have a point $x_0 \in X$ such that $x_0 \in T^-(x_0)$ or $x_0 \in T(x_0)$. This contradicts (3). This completes our proof.

Note that, in case $X = D$, Theorem 4.3 reduces to a maximal element theorem.

Moreover, from Theorem 4.1, we have the following minimal element theorem:

Theorem 4.4. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ two maps satisfying*

(1) *for each $z \in D$, $S(z)$ is open [resp. closed];*

(2) *for each $y \in X$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$;*

(3) *$T(X) \subset S(N)$ for some $N \in \langle D \rangle$; and*

(4) *$x \notin T(x)$ for all $x \in X$.*

Then T has a minimal element $x_* \in X$; that is, $T^-(x_*) = \emptyset$.

Proof. Suppose contrary that $T^-(x) \neq \emptyset$ for each $x \in X$. Then T is surjective, that is, $X = T(X) = S(N)$. Hence, by Theorem 4.1, T has a fixed point. This contradicts (4).

5. Almost fixed points

A G-convex space $(X \supset D; \Gamma)$ is called an *LG-space* (or a *locally G-convex space*) if $X = (X, \mathcal{U})$ is a Hausdorff uniform space such that D is dense in X and if there exists a basis $\{V_\lambda\}_{\lambda \in I}$ for the uniformity $\mathcal{U}$ such that for each $\lambda \in I$, $\{x \in X \mid C \cap V_\lambda[x] \neq \emptyset\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, where

$$V_\lambda[x] = \{x' \in X \mid (x, x') \in V_\lambda\}.$$

For a C-space $(X; \Gamma)$, an LG-space reduces to an LC-space [8,9] (or a locally C-convex space [26]). Any nonempty convex subset X of a locally convex Hausdorff t.v.s. E is an obvious example of an LC-space $(X; \Gamma)$ with $\Gamma_A = \mathrm{co}\, A$ for $A \in \langle X \rangle$. For other examples, see [8,26].

A G-convex space $(X; \Gamma)$ is called an *LG-metric space* if X is equipped with a metric d such that for any $\varepsilon > 0$, the set $\{x \in X \mid d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex and open balls are Γ-convex. This notion generalizes that of LC-metric spaces due to Horvath [8].

From the KKM Theorem 2.1, we obtain the following almost fixed point theorem for u.s.c. or l.s.c. multimaps on LG-spaces:

Theorem 5.1. *Let $(X \supset D; \Gamma)$ be an LG-space and $T : X \multimap X$ an u.s.c. [resp., a l.s.c.] multimap with Γ-convex values such that $T(X)$ is totally bounded. Then T has the almost fixed point property [that is, for each $V \in \{V_\lambda\}_{\lambda \in I}$, there exists an $x_V \in X$ such that $T(x_V) \cap V[x_V] \neq \emptyset$].*

Proof. We may assume that V_λ is always closed [resp. open] for $\lambda \in I$. Let $V \in \{V_\lambda\}_{\lambda \in I}$. Then there exists an open member W of $\mathcal{U}$ such that $W \subset V$. Note that for each $x \in X$, $W[x]$ is an open neighborhood of x. Since $K := \overline{T(X)}$ is totally bounded and D is dense in X, there exists an $M := \{y_1, \ldots, y_n\} \in \langle D \rangle$ such that $K \subset \bigcup_{y \in M} W[y]$.

For each $y_i \in M$, let $F(y_i) := \{x \in X \mid T(x) \cap V[y_i] = \emptyset\}$. Since T is u.s.c. [resp. l.s.c.], each $F(y_i)$ is open [resp. closed]. Moreover, since $T(X) \subset K \subset \bigcup_{i=1}^n V[y_i]$, we have

$$\bigcap_{i=1}^{n} F(y_i) \subset \{x \in X \mid T(x) \cap \bigcup_{i=1}^{n} V[y_i] = \emptyset\} = \emptyset.$$

We will apply Theorem 2.1 to the G-convex space $(X \supset M; \Gamma)$. Since the conclusion of Theorem 2.1 does not hold, $F : M \multimap X$ can not be a KKM map; that is, there exist an $N \in \langle M \rangle$ and an $x_V \in \Gamma_N$ such that $x_V \notin F(N) = \bigcup_{y \in N} F(y)$. Hence $T(x_V) \cap V[y] \neq \emptyset$ for all $y \in N$, and

$$N \subset L := \{y \in X \mid T(x_V) \cap V[y] \neq \emptyset\}.$$

Since $T(x_V)$ is Γ-convex and $(X \supset D; \Gamma)$ is an LG-space, L is Γ-convex. Therefore, $x_V \in \Gamma_N \subset L$ and hence $T(x_V) \cap V[x_V] \neq \emptyset$.

From Theorem 5.1, we can deduce the following fixed point theorem in [23]:

Theorem 5.2. *Let $(X \supset D; \Gamma)$ be an LG-space and $T : X \multimap X$ a compact u.s.c. multimap with closed Γ-convex values. Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Proof. For each basis element V, there exist $x_V, y_V \in X$ such that $y_V \in T(x_V)$ and $y_V \in V[x_V]$. Since $T(X)$ is relatively compact, we may assume that the net y_V converges to some $x_0 \in K$. Then x_V also converges to x_0. Since T is u.s.c. with closed values, the graph of T is closed in $X \times \overline{T(X)}$, and hence we have $x_0 \in T(x_0)$. This completes our proof.

In [23], Theorem 5.2 is applied to various fixed point theorems for LG-spaces, LC-spaces, hyperconvex spaces, and normed vector spaces. For example, we have the following:

Corollary (Himmelberg)**.** *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $T : X \multimap X$ a compact u.s.c. multimap with closed convex values. Then T has a fixed point $x_0 \in T(x_0)$.*

Recall that the Himmelberg theorem unifies and generalizes historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, and others. For the literature, see [17].

For convex subsets of a t.v.s., from Theorem 3.1, we have the following almost fixed point theorem for the class $\mathfrak{KC}$:

Theorem 5.3. *Let X be a convex subset of a t.v.s. E and $F \in \mathfrak{KC}(X, X)$ such that $F(X)$ is totally bounded. Then for any convex neighborhood V of 0 in E, there exists an $x_* \in X$ such that $F(x_*) \cap (x_* + V) \neq \emptyset$.*

Proof. Define $S = T : X \multimap X$ by $T(x) = (x + V) \cap X$ for each $x \in X$. We may assume that V is open, convex, and symmetric. Then $T(x)$ is open for each $x \in X$. Moreover, for each $y \in X$,

$$T^-(y) = \{x \in X \mid y \in (x + V) \cap X\} = \{x \in X \mid x \in y - V\}$$

is convex. Since $F(X)$ is totally bounded, there exists a subset $\{x_1, x_2, \cdots, x_n\} \subset X$ such that

$$\overline{F(X)} \subset \bigcup_{i=1}^{n} (x_i + V) \cap X = \bigcup_{i=1}^{n} T(x_i).$$

Therefore, by Theorem 3.1, the conclusion follows.

Similarly, from Theorem 3.1′, we can obtain the following for the class $\mathfrak{KO}$:

Theorem 5.3′. *Let X be a totally bounded convex subset of a t.v.s. E and $F \in \mathfrak{KO}(X, X)$. Then for each closed convex neighborhood V of 0 in E, there exists an $x_* \in X$ such that $F(x_*) \cap (x_* + V) \neq \emptyset$.*

From Theorem 5.3, we immediately have the following as in the proof of Theorem 5.2:

Theorem 5.4. (Chang–Yen [4]) *Let X be a convex subset of a locally convex Hausdorff t.v.s. E. Then any closed compact map $F \in \mathfrak{KC}(X, X)$ has a fixed point.*

A *convex space* X (in the sense of Lassonde) is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*.

Let X and Y be topological spaces. An *admissible class* $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K, Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact-valued; and

(iii) for each polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, σ-selectional maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park *et al.*, and approximable maps $\mathbb{A}^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of $\mathfrak{A}_c^\kappa$. For the literature, see [13,14,17].

We now define a "better" admissible class of maps defined on a convex space X:

$$F \in \mathfrak{B}(X, Y) \iff \text{for any polytope } P \text{ in } X \text{ and any } f \in \mathbb{C}(F(P), P),$$
$$f(F|_P) : P \multimap P \text{ has a fixed point.}$$

Note that $\mathfrak{A}_c^\kappa \subset \mathfrak{B}$ and some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were given in [15].

The following is known [14,15]:

Lemma. *Let X be a convex space and Y a Hausdorff space. Then*

(1) $\mathfrak{A}_c^\kappa(X,Y) \subset \mathfrak{KC}(X,Y)$; *and*
(2) *in the class of closed compact multimaps, two classes $\mathfrak{B}(X,Y)$ and $\mathfrak{KC}(X,Y)$ coincide.*

It should be noted that there have been found only a few trivial examples of maps in $\mathfrak{KC}(X,Y)$ which are not in $\mathfrak{A}_c^\kappa$ or $\mathfrak{B}$; see [4].

From Theorem 5.4 and Lemma, we have the following:

Theorem 5.5. *Let X be a convex subset of a locally convex Hausdorff t.v.s. E. Then any closed compact map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Later, we found that Theorem 5.5 holds for an admissible (in the sense of Klee) convex subset X of a Hausdorff t.v.s. E; see [17].

6. Minimax theorems

In this section, Z denotes a *complete linearly ordered space*, that is, a linearly ordered set whose every subset has a least upper bound. Examples are the extended real line $\overline{\mathbf{R}}$, the extended Euclidean space $\overline{\mathbf{R}}^n$, and any compact (in the Euclidean topology) subset of $\mathbf{R}^n$ with respect to the lexicographic order; see Komornik [11]. Our aim in this section is to show that the results in our previous work [18] can be extended to G-convex spaces.

For a topological space X, a function $f : X \to Z$ is said to be *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] whenever $\{x \in X \mid f(x) > z\}$ [resp. $\{x \in X \mid f(x) < z\}$] is open in X for each $z \in Z$.

If X is compact and $f : X \to Z$ is l.s.c., then there exists an $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. For any family $\{f_i\}_{i\in I}$ of l.s.c. functions $f_i : X \to Z$, the function $\sup_{i\in I} f_i$ is also l.s.c.; see [11].

The following is the main result of this section:

Theorem 6.1. *Let $(X;\Gamma)$ be a G-convex space, $(Y;\Gamma')$ a Hausdorff compact G-convex space, and $f : X \times Y \to Z$ a function. Suppose that*

(1) *there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap (a, b) \neq \emptyset$;*
(2) *$f(x, \cdot)$ is l.s.c. on Y and $\{y \in Y \mid f(x, y) < s\}$ is Γ'-convex for each $x \in X$ and $s \in U$; and*
(3) *$f(\cdot, y)$ is u.s.c. on X and $\{x \in X \mid f(x, y) > s\}$ is Γ-convex for each $y \in Y$ and $s \in U$.*

Then

$$\sup_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \sup_{x\in X} f(x, y).$$

Proof. Since $f(x, \cdot)$ is l.s.c., $p(x) := \min_{y\in Y} f(x, y)$ exists for each $x \in X$. Since $q(y) := \sup_{x\in X} f(x, y)$ is l.s.c. for each $y \in Y$, $q(y_0) = \min_{y\in Y} q(y)$ exists. Note that

$$p(x) = \min_{y\in Y} f(x, y) \le f(x, y) \le \sup_{x\in X} f(x, y) = q(y)$$

for all $x \in X$ and $y \in Y$. Therefore, we have

$$\sup_{x\in X} p(x) \le \min_{y\in Y} q(y).$$

Suppose that the equality does not hold. Then there exists an $s \in U$ such that

$$\sup_{x\in X} p(x) < s < \min_{y\in Y} q(y).$$

We define multimaps $F, T : X \multimap Y$ by

$$F(x) := \{y \in Y \mid f(x, y) < s\} \text{ and } T(x) := \{y \in Y \mid f(x, y) > s\}$$

for $x \in X$. Then $F(x)$ is nonempty and Γ'-convex by (2), and $T(x)$ is open since $f(x, \cdot)$ is l.s.c. Moreover,

$$F^{-}(y) = \{x \in X \mid f(x, y) < s\} \text{ and } T^{-}(y) = \{x \in X \mid f(x, y) > s\}$$

for $y \in Y$. Then $F^{-}(y)$ is open since $f(\cdot, y)$ is u.s.c., and $T^{-}(y)$ is nonempty and Γ'-convex by (3). Then it is known that $F \in \mathfrak{A}_c^{\kappa}(X, Y) \subset \mathfrak{KC}(X, Y)$. Now, by applying Theorem 3.1, there exists an $x_0 \in X$ such that $F(x_0) \cap T(x_0) \neq \emptyset$. This contradicts

$$f(x_0, a) < s < f(x_0, b) \quad \text{for each } a \in F(x_0) \text{ and } b \in T(x_0).$$

This completes our proof.

Remark. For convex spaces, if $U = Z$, then Theorem 6.1 is a consequence of Komornik [11, Theorem 2] for interval spaces with different proof. For convex spaces, Theorem 6.1 reduces to Park [18, Theorem 3] which includes a lot of known results.

Corollary. *Under the hypothesis of Theorem 6.1, further if X is compcat, then f has a saddle point.*

Proof. Since $f(x, \cdot)$ and $f(\cdot, y)$ are l.s.c. and u.s.c., resp., $p(x) = \min_{y \in Y} f(x, y)$ and $q(y) = \max_{x \in X} f(x, y)$ exist for each $x \in X$ and $y \in Y$. Since p is u.s.c. on X and q is l.s.c. on Y, $\max_{x \in X} p(x) = p(x_0)$ and $\min_{y \in Y} q(y) = q(y_0)$ for some $x_0 \in X$ and $y_0 \in Y$. Then (x_0, y_0) is a saddle point by Theorem 6.1. This completes our proof.

Remark. For convex spaces, Corollary reduces to Arandjelović [1, Theorem 3], which extends the Sion minimax theorem; see [18].

The following new minimax theorem is a variant of Theorem 6.1:

Theorem 6.2. *Let $(X;\Gamma)$ be a G-convex space, Y a Hausdorff compact space, and $f : X \times Y \to Z$ a l.s.c. function such that*

(1) *there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap [a, b) \neq \emptyset$;*
(2) *for each $s \in U$ and $y \in Y$, $\{x \in X \mid f(x, y) > s\}$ is Γ-convex; and*
(3) *for each $s \in U$ and $x \in X$, $\{y \in X \mid f(x, y) \leq s\}$ is acyclic.*

Then

$$\sup_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \sup_{x \in X} f(x, y).$$

Proof. As in the proof of Theorem 6.1, we have

$$\sup_{x \in X} p(x) \leq \min_{y \in Y} q(y).$$

Suppose that the equality does not hold. Then there exists an $s \in U$ such that

$$\sup_{x \in X} p(x) \leq s < \min_{y \in Y} q(y).$$

We define multimaps $F, T : X \multimap Y$ by

$$F(x) := \{y \in Y \mid f(x, y) \leq s\} \text{ and } T(x) := \{y \in Y \mid f(x, y) > s\}$$

for $x \in X$. Then $F(x)$ is nonempty by the definition of $F(x)$ and closed since $f(x, \cdot)$ is l.s.c. for each $x \in X$. On the other hand, $T(x)$ is open since $f(x, \cdot)$ is l.s.c. Moreover, for each $y \in Y$,

$$T^{-}(y) = \{x \in X \mid f(x, y) > s\}$$

is nonempty and Γ-convex by (2).

Consider the graph of F

$$\text{Gr}(F) = \{(x, y) \in X \times Y \mid f(x, y) \leq s\}.$$

Since f is l.s.c., $\mathrm{Gr}(F)$ is closed in $X \times Y$. Since Y is compact, F is u.s.c. Note that each $F(x)$ is acyclic by (3). Hence F is an acyclic map and hence $F \in \mathfrak{KC}(X, Y)$.

Therefore by Theorem 3.1, there exists an $x_0 \in X$ such that $F(x_0) \cap T(x_0) \neq \emptyset$. This leads a contradiction as in the proof of Theorem 6.1.

Remark. For convex spaces, related results were obtained by Ha [7, Theroem 4], Komornik [11, Theorem 3], and Komiya [10, Theorem 3]. For convex spaces, Theorem 6.2 reduces to Park [18, Theorem 4].

7. Extensions of monotone sets

Given two t.v.s. E and F, let $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ be a bilinear pairing which is continuous on compact subsets of $F \times E$. This assumption is quite natural in most applications, since the natural pairing between a Hausdorff locally convex t.v.s. E and its dual space E^* equipped with the strong topology enjoys this property.

A subset $M \subset E \times F$ is said to be *monotone* if for any two points (u, w) and (u', w') in M, we have $\langle w - w', u - u' \rangle \geq 0$; see Debrunner and Flor [5].

Browder [3] obtained the following extension theorem for monotone sets:

Lemma. (Browder) *Let K be a compact convex subset of a t.v.s. E, and F a t.v.s. with a bilinear pairing $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ which is continuous on compact subsets of $F \times E$. Let $f : K \to F$ be continuous and M a monotone subset of $K \times F$. Then there exists a $u_0 \in K$ such that*

$$\langle f(u_0) - w, u_0 - u \rangle \geq 0 \text{ for all } (u, w) \in M,$$

or equivalently, the set $M \cup \{(u_0, f(u_0))\}$ remains monotone.

This result sharpens corresponding result of Debrunner and Flor [5] for E locally convex and of Fan [6, Theorem 12] for F locally convex and quasi-complete.

We give slightly extended versions of results in [24].

We deduce the following equilibrium existence theorem from Theorem 3.1:

Theorem 7.1. *Let K be a compact convex subset of a t.v.s. E, K_1 a compact subset of a t.v.s. F, $A \in \mathfrak{KC}(K, K_1)$ with closed graph, and $M \subset E \times F$. Let $\Phi : E \times F \to \mathbf{R} \cup \{-\infty\}$ be a function such that*

(1) *Φ is u.s.c. on compact subsets of $E \times F$;*

(2) *for each $x \in E$, $\Phi(x, \cdot)$ is l.s.c. on compact subsets of F; and*

(3) *for each $w \in F$, $\Phi(\cdot, w)$ is quasiconcave.*

Suppose that for each $y \in K_1$, there exists an $x \in K$ such that

$$\Phi(x - u, y - w) \geq 0 \quad \text{for all } (u, w) \in M.$$

Then there exist a $u_0 \in K$ and a $w_0 \in A(u_0)$ such that

$$\Phi(u_0 - u, w_0 - w) \geq 0 \quad \text{for all } (u, w) \in M.$$

Proof. For any $\varepsilon > 0$ and any nonempty finite subset N of M, we set

$$H_{(\varepsilon,N)} = \{(u_0, w_0) \in \mathrm{Gr}(A) \,|\, \Phi(u_0 - u, w_0 - w) \geq -\varepsilon \text{ for all } (u, w) \in N\}$$

and

$$\begin{aligned} H_0 &= \{(u_0, w_0) \in \mathrm{Gr}(A) \,|\, \Phi(u_0 - u, w_0 - w) \geq 0 \text{ for all } (u, w) \in M\} \\ &= \bigcap \{H_{(\varepsilon,N)} \,|\, \varepsilon > 0 \text{ and } N \text{ is a finite subset of } M\}. \end{aligned}$$

Then we have to show $H_0 \neq \emptyset$.

By (1), each $H_{(\varepsilon,N)}$ is a closed subset of $\mathrm{Gr}(T)$. The intersection of each finite family of such sets is also a set of the form $H_{(\varepsilon',N')}$ for some $\varepsilon' > 0$ and a finite subset N' of M. Therefore, in order to show $H_0 \neq \emptyset$, it suffices to show that each $H_{(\varepsilon,N)}$ is nonempty.

Choose a given $\varepsilon > 0$ and a nonempty finite subset N of M. Define a map $T : K \multimap K_1$ by

$$T(x) = \{y \in K_1 \,|\, \Phi(x - u, y - w) > -\varepsilon, \ (u, w) \in N\}$$

for $x \in X$. Then $T(x)$ is open in K_1 by (2). Moreover,

$$T^{-}(y) = \{x \in K \mid \Phi(x-u, y-w) > -\varepsilon,\ (u,w) \in N\}$$

is nonempty by hypothesis and convex by (3).

Now we apply Theorem 3.1. Then there exists a $(u_0, w_0) \in \mathrm{Gr}(A)$ such that $w_0 \in T(u_0)$; that is,

$$\Phi(u_0 - u, w_0 - w) > -\varepsilon \quad \text{for all } (u,w) \in N.$$

Therefore, $H_{(\varepsilon,N)}$ is nonempty. This completes our proof.

Remarks. 1. For the subclass $\mathbb{C}$ of $\mathfrak{KC}$, Theorem 7.1 reduces to Fan [6, Theorem 11], who assumed that F is locally convex and other restrictions.

2. For the subclass $\mathbb{K}$ of $\mathfrak{KC}$, Theorem 7.1 reduces to Browder [3, Theorem 9], where F is locally convex.

The following is our theorem on extensions of monotone sets:

Theorem 7.2. *Let E be a t.v.s., F a t.v.s. with a bilinear pairing $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ which is continuous on compact subsets of $F \times E$, K a compact convex subset of E, and K_1 a compact subset of F. Let $A \in \mathfrak{KC}(K, K_1)$ have closed graph and M a monotone subset of $K \times F$. Then there exist a $u_0 \in K$ and a $w_0 \in A(u_0)$ such that*

$$\langle w_0 - w, u_0 - u\rangle \geq 0 \quad \textit{for all } (u,w) \in M.$$

Proof. We put $\Phi(x,w) = \langle w, x\rangle$ for $(x,w) \in E \times F$. Then Φ satisfies conditions (1)–(3) in Theorem 7.1. By Theorem 7.1, it suffices to show that for each $y \in K_1$, there exists an $x \in K$ such that

$$\langle y - w, x - u\rangle \geq 0 \quad \text{for all } (u,w) \in M.$$

Now, we define $f : K \to K_1$ by

$$f(v) = y \quad \text{for all } v \in K.$$

By applying Lemma to f, such an $x \in K$ exists. This completes our proof.

Remarks. 1. For the subclass $\mathbb{C}$ of $\mathfrak{KC}$, Theorem 7.2 reduces to Browder [3, Theorem 8] or Lemma.

2. Even for the subclass $\mathbb{K}$ of $\mathfrak{KC}$, Theorem 7.2 improves Browder [3, Theorem 9], where F is assumed to be locally convex.

References

[1] I. Arandjelović, *An extension of the Sion's minimax theorem*, Zb. rad. Fil. fak. u Nišu, Ser. Mat. **6** (1992), 1–3.

[2] A. Borglin and H. Keiding, *Existence of equilibrium actions and of equilibrium*, J. Math. Econom. **3** (1976), 313–316.

[3] F.E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–310.

[4] T.-H. Chang and C.-L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl. **203** (1996), 224–235.

[5] H. Debrunner und P. Flor, *Ein Erweiterungssatz für monotone Mengen*, Arch. Math. **15** (1964), 445–447.

[6] Ky Fan, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203.

[7] C.W. Ha, *Minimax and fixed point theorems*, Math. Ann. **248**(1980), 73–77.

[8] C.D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.

[9] ______, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2** (1993), 253–269.

[10] H. Komiya, *Coincidence theorem and saddle point theorem*, Proc. Amer. Math. Soc. **96** (1986), 599–602.

[11] V. Komornik, *Minimax theorems for upper semicontinuous functions*, Acta Math. Acad. Sci. Hungar. **40** (1982), 159–163.

[12] Sehie Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, ed.), World Sci., River Edge, NJ, 1992, pp. 248–277.

[13] ______, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc. **31** (1994), 493–519.

[14] ______, *Coincidence theorems for the better admissible multimaps and their applications*, Proc. WCNA 96, Nonlinear Anal. **30** (1997), 4183-4191.

[15] ______, *Remarks on fixed point theorems of Ricceri*, Proc. '97 Workshop on Math. Anal. & Appl., Pusan Nat. Univ., 1997, pp.1–9.

[16] ______, *A generalized minimax inequality related to admissible multimaps and its applications*, J. Korean Math. Soc. **34** (1997), 719–730.

[17] ______, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829. *Corrections*, ibid. **36** (1999), 829–832.

[18] ______, *Minimax theorems in convex spaces*, Novi Sad J. Math. **28** (1998), 1–8.
[19] ______, *Minimax theorems and the Nash equilibria on generalized convex spaces*, Josai Math. Monograph **1** (1999), 33–46.
[20] ______, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comp. Appl. Math. **7** (2000), 1–28.
[21] ______, *Remarks on topologies of generalized convex spaces*, Nonlinear Funct. Anal. Appl. **5** (2000), 67–79.
[22] ______, *New topological versions of the Fan–Browder fixed point theorem*, Nonlinear Anal. **47** (2001), 595-606.
[23] ______, *Fixed point theorems in locally G-convex spaces*, Nonlinear Anal. (2001).
[24] ______, *Extensions of monotone sets*, Set Valued Mappings with Applications in Nonlinear Analysis (R.P. Agarwal and D. O'Regan, eds.), Gordon and Breach Publ., 2001, pp.403–409.
[25] S. Park and W. Lee, *A unified approach to generalized KKM maps in generalized convex spaces*, J. Nonlinear and Convex Analysis **2** (2001), 157–166.
[26] E. Tarafdar, *Fixed point theorems in locally H-convex uniform spaces*, Nonlinear Anal. **29** (1997), 971–978.
[27] N. Yannelis and N. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Economics **12** (1983), 233–245.

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