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Remarks on generalized quasi-equilibrium problems☆

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Abstract

Our main aim in this paper is to obtain fundamental existence theorems of quasi-equilibrium problems and generalized quasi-equilibrium problems of multivalued functions. We use a continuous selection theorem to show that the existence theorems of generalized quasi-equilibrium problems are simple consequences of the corresponding quasi-equilibrium problems. We use a generalization of Himmelberg's theorem due to the third author for not necessary locally convex topological vector spaces to prove the existence theorems of quasi-equilibrium problems for multivalued functions.

Keywords: Quasi-equilibrium; Upper semicontinuous; Compact multimap; Acyclic map; Local intersection property; Closed map

1. Introduction

By an *equilibrium problem* (shortly, EP), Blum and Oettli [2] understood the problem of finding

$$\hat{x} \in X \quad \text{such that } \varphi(\hat{x}, y) \leqslant 0 \text{ for all } y \in X,$$

where X is a given set and $\varphi : X \times X \to \bar{R}$ is a given extended real valued function with $\varphi(x,x) = 0$ for all $x \in X$. They observed that the problem contains as special cases for

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instance optimization problems, problems of the Nash type equilibria, complementarity problems, fixed point problems, variational inequalities, and many others.

Moreover, some variations or generalizations of this problem can be possible; see Noor and Oettli [10] and references therein. Motivated by such situation, we introduce the following generalized EP:

A *quasi-equilibrium problem* (shortly, QEP) is to find

$$\hat{x} \in S(\hat{x}) \quad \text{such that } \varphi(\hat{x}, z) \leqslant 0 \text{ for all } z \in S(\hat{x}),$$

where X is a given set, $S : X \multimap X$ a given multimap with nonempty values, and $\varphi : X \times X \to \bar{R}$ a given function.

A *general quasi-equilibrium problem* (shortly, GQEP) is to find

$$\hat{x} \in S(\hat{x}) \text{ and } \hat{y} \in T(\hat{x}) \quad \text{such that } \varphi(\hat{x}, \hat{y}, z) \leqslant 0 \quad \text{for all } z \in S(\hat{x}),$$

where X and S are the same as above, Y another given set, $T : X \multimap Y$ another multimap, and $\varphi : X \times Y \times X \to \bar{R}$ a given function.

Our main aim in this paper is to obtain fundamental existence theorems for solutions of QEPs or GQEPs in more general situation where the single extended real valued function φ is replaced by a multi-valued function $\Phi : X \times X \multimap \bar{R}$ or $\Phi : X \times Y \times X \multimap \bar{R}$.

Recently, Wu and Shen [15] claimed that they obtained a further generalization of the Yannelis–Prabhakar selection theorem [16] and then, applied it to several results on GQEPs. Note that the selection theorem was also due to Ben-El-Mechaiekh et al. [1] in the same year. Main feature of their results [15] is that firstly, the map T in their GQEPs has a continuous selection and, secondly, their main tool is Himmelberg's fixed point theorem for locally convex topological vector spaces. (t.v.s.) [4].

In this paper, we show that their selection theorem is not new, that GQEPs are simple consequences of corresponding QEPs whenever the map T has a continuous selection, that the function $\varphi : X \times X \to \bar{R}$ in QEPs with $\varphi(x, x) = 0$ for all $x \in X$ can be replaced by $\Phi : X \times X \multimap \bar{R}$ with $\Phi(x, x) \leqslant 0$ for all $x \in X$, and that the function $\varphi : X \times Y \times X \to \bar{R}$ in GQEPs with $\varphi(x, y, x) = 0$ for all $(x, y) \in X \times Y$ can be replaced by $\Phi : X \times Y \times X \multimap \bar{R}$ with $\Phi(x, y, x) \leqslant 0$ for all $(x, y) \in X \times Y$. Moreover, we use a generalization of Himmelberg's theorem due to the third author [12] for not necessarily locally convex topological vector spaces. Consequently, we can drastically generalize and improve all of the results in [15] and some others.

2. Preliminaries

A *multimap* or a *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values $T(x)$ for $x \in X$ and fibers $T^{-}(y)$ for $y \in Y$. Note that $x \in T^{-}(y)$ if and only if $y \in T(x)$.

Recall that an extended real valued function $g : X \to \bar{R}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : g(x) > r\}$ [resp. $\{x \in X : g(x) < r\}$] is *open* for each $r \in \bar{R}$. If X is a convex set in a vector space, then $g : X \to \bar{R}$ is *quasi-concave* [resp. *quasi-convex*] if $\{x \in X : g(x) > r\}$ [resp. $\{x \in X : g(x) < r\}$] is convex for each $r \in \bar{R}$.

Throughout this paper, all topological spaces are assumed to be Hausdorff, t.v.s., and co denotes the convex hull.

For topological spaces X and Y, a map $T:X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T)=\{(x,y):x\in X, y\in T(x)\}$ is closed in $X\times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A map $T:X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B\subset Y$, the set $T^-(B)=\{x\in X: T(x)\cap B\neq\emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B\subset Y$, the set $T^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c.

The following are well known:

Lemma 1. *Let X and Y be two topological spaces. If $T:X\multimap Y$ is u.s.c. with closed values, then T is closed.*

Lemma 2. *Let X be a topological space and Y a compact space. If $T:X\multimap Y$ is closed, then T is u.s.c.*

We denote $R^+=\{x\in R: 0\leqslant x<\infty\}$. For $A\subset R$ and $a\in R$, we denote $A\leqslant a$ if $x\leqslant a$ for all $x\in A$.

Lemma 3 (Tan and Yuan [14]). *Let X and Y be topological spaces, A an open subset of X, and $F_1:X\multimap Y$, $F_2:A\multimap Y$ u.s.c. maps such that $F_2(x)\subset F_1(x)$ for all $x\in A$. Then the map $F:X\multimap Y$ defined by*

$$F(x)=\begin{cases} F_1(x) & \text{if } x\notin A,\\ F_2(x) & \text{if } x\in A\end{cases}$$

is also u.s.c.

The quasi-convexity of a single-valued function can be extended to multi-valued function as follows:

For a convex set X, a multimap $G:X\multimap \bar{R}$ is said to be *quasi-convex* if for any $\lambda\in\bar{R}$, the set $\{x\in X:$ there is a $y\in G(x)$ such that $y\leqslant\lambda\}$ is convex.

Lemma 4 (Lin [8]). *Let X be a convex set. Then $G:X\multimap\bar{R}$ is quasi-convex if and only if for any $x_1,x_2\in X, 0\leqslant\lambda\leqslant 1$, $y_1\in G(x_1)$ and $y_2\in G(x_2)$, there exists $y\in G[\lambda x_1+(1-\lambda)x_2]$ such that $y\leqslant\max\{y_1,y_2\}$.*

Lemma 5 (Generalized Berge theorem [9]). *Let X and Y be topological spaces, $G:X\times Y\multimap\bar{R}$, $S:X\multimap Y$, $m(x)=\sup_{y\in S(x)}G(x,y)$, and $M(x)=\{y\in S(x): m(x)\in G(x,y)\}$*

(a) *If S is l.s.c. and G is l.s.c., then m is l.s.c.*

(b) *If both G and S are u.s.c. with compact values, then m is u.s.c.*

(c) *If both G and S are continuous with compact values, then m is a continuous function and M is a closed u.s.c. map.*

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [6]) provided that, for every compact subset A of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h: A \to X$ such that $x - h(x) \in V$ for all $x \in A$ and $h(A)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other example of admissible t.v.s. are l^p and $L^p(0,1)$ for $0 < p < 1$, the Hardy space H^p for $0 < p < 1$, certain Orlicz spaces, and many others. For details, see Park [12] and references therein.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any convex or star-shaped subset of a t.v.s is contractible, and any contractible space is acyclic. A map $T: X \multimap Y$ is said to be *acyclic* if it is u.s.c. with acyclic compact values.

Lemma 6 (Park [12]). *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact acyclic map $G: X \multimap X$ has a fixed point $x \in X$; that is $x \in G(x)$.*

Lemma 7 (Park [11]). *Let K be a compact convex subset of a real t.v.s. on which its topological dual E^* separates points, and $S: K \multimap K$ a multimap with closed convex values such that*

(1) *for each $p \in E^*$, $\{x \in K: \sup p(S(x)) \geqslant p(x)\}$ is closed.*

Let $f: K \times K \to \bar{R}$ satisfy the following:

(2) *for each $y \in K, x \mapsto f(x,y)$ is l.s.c.; and*

(3) *$f(x,y)$ is 0-DCV in y; that is, for each finite subset $\{y_1, y_2, \ldots, y_m\}$ of K and each $y_0 = \sum_{i=1}^m \alpha_i y_i (\alpha_i \geqslant 0, \sum_{i=1}^m \alpha_i = 1)$, we have*

$$\sum_{i=1}^{m} \alpha_i f(y_0, y_i) \geqslant 0.$$

Suppose that S and f are related by

(4) *$\{x \in K: \sup_{y \in S(x)} f(x,y) \leqslant 0\}$ is closed. Then there exists a solution $\bar{x} \in K$ to the quasi-variational inequality:*

$$\bar{x} \in S(\bar{x}) \quad \textit{and} \quad \sup_{y \in S(\bar{x})} f(\bar{x}, y) \leqslant 0.$$

3. Selection theorems

In 1991, Horvath [5, Theorem 3.2] obtained a continuous selection theorem for his c-spaces or the so-called H-spaces. The convex space version of his theorem is as follows:

Theorem 1 (Horvath [5]). *Let X be a paracompact space, Y a convex space, and $S, T: X \multimap Y$ two maps such that*

(i) *for any $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and*

(ii) *$X = \bigcup\{\operatorname{Int} S^-(y): y \in Y\}$.*

Then T has a continuous selection $g: X \to Y$; that is, $g(x) \in T(x)$ for each $x \in X$.

Note that any convex subset of a t.v.s. is a convex space (in this sense of Lassonde) [7], and not conversely. Moreover, Theorem 1 extends results due to Ben-El-Mechaiekh et al. [1] and Yannelis and Prabhakar [16].

Since Horvath's theorem appeared, several authors claimed to obtain generalizations of theorems of [1,16]; for example, see [3,15]. Especially, Wu and Shen [15] introduced the following concept and a simple consequence of Theorem 1 as follows:

For topological spaces X and Y, a map $T: X \multimap Y$ is said to have the local intersection property if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T(z) \neq \emptyset$.

Corollary 2 (Wu and Shen [15]). *Let X be a paracompact space E and Y a nonempty subset of a t.v.s. F. Suppose that $S, T: X \multimap Y$ are two maps such that*

(i) *for each $x \in X$, $S(x)$ is nonempty and* $\operatorname{co} S(x) \subset T(x)$.

(ii) *S has local intersection property.*

Then T has a continuous selection.

Proof. It suffices to show that (ii) implies $X = \bigcup_{y \in Y} \operatorname{Int} S^{-}(y)$. For any $x \in X$, there exists an open neighborhood $N(x)$ such that $M(x) := \bigcap_{z \in N(x)} S(z) \neq \emptyset$. For $y \in M(x)$, we have $y \in S(z)$ for all $z \in N(x)$ and hence $z \in S^{-}(y)$. Therefore $x \in N(x) \subset S^{-}(y)$. Since $N(x)$ is open, we have $x \in N(x) \subset \operatorname{Int} S^{-}(y)$. This implies $X = \bigcup_{y \in Y} \operatorname{Int} S^{-}(y)$.

Remark. If $X = \bigcup_{y \in Y} \operatorname{Int} S^{-}(y)$, then $S(x) \neq \emptyset$ for all $x \in X$. Indeed, for all $x \in X$, $x \in \operatorname{Int} S^{-}(y)$ for some $y \in Y$. Hence $x \in S^{-}(y)$ and $y \in S(x) \neq \emptyset$.

Example (Wu and Shen [15]). Let $E = F = R$, $X = Y = [0,2)$ and $S(x) = [x,2)$ for each $x \in X$. Then $S^{-}(y) = [0,y]$ is not open is X and $\operatorname{Int} S^{-}(y) = [0,y)$ and $X = \bigcup_{y \in Y} \operatorname{Int} S^{-}(y)$. Thus Theorem 1 is indeed a proper generalization of Ben-El-Mechaiekh et al. [1] and Yannelis and Prabhakar [15].

From Theorem 1, we have the following fixed point theorem which improves [15, Corollary 3]:

Corollary 3. *Let X be a paracompact admissible convex subset of a t.v.s., K a nonempty compact subset of X, and $S, T: X \multimap K$ two maps such that*

(i) *for any $x \in X$,* $\operatorname{co} S(x) \subset T(x)$; *and*

(ii) $X = \bigcup \{ \operatorname{Int} S^{-}(y) : y \in X \}$

Then T has a fixed point $\bar{x} \in K$; that is, $\bar{x} \in T(\bar{x})$.

Proof. By Theorem 1, T has a continuous selection $f: X \to K \subset \operatorname{co} K \subset X$. Since f is compact, by Lemma 6, f has a fixed point $\bar{x} \in X$ and hence $\bar{x} = f(\bar{x}) \in T(\bar{x})$. This completes our proof. □

4. Quasi-equilibrium problems

In this section, we begin with the following existence theorem for solutions of QEPs with respect to a multifunction $\Phi : X \times X \multimap \bar{R}$.

Theorem 4. *Let X be an admissible convex subset of a t.v.s. E, $S : X \multimap X$ a compact closed map, and $\Phi : X \times X \multimap \bar{R}$ an u.s.c. map with compact values. Suppose that*
(1) *the function m defined on X by*

$$m(x) = \sup_{u \in S(x)} \Phi(x,u) \quad for\ x \in X$$

is l.s.c.; and
(2) *for each $x \in X$, the set*

$$M(x) = \{y \in S(x) : m(x) \in \Phi(x, y)\}$$

is acyclic.

Then there exists an $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$ and $m(\hat{x}) \in \Phi(\hat{x},\hat{x})$. Further, if $\Phi(x,x) \leqslant 0$ for all $x \in X$, then there exists an $\hat{x} \in S(\hat{x})$ such that

$$\Phi(\hat{x}, y) \leqslant 0 \quad for\ all\ y \in S(\hat{x}).$$

Proof. Since S is a compact closed map, it follows from Lemma 2 that S is u.s.c. with compact values. Since Φ is an u.s.c. map with compact values, it follows from Lemma 5 that m is u.s.c. By assumption m is l.s.c., and hence m is continuous. We claim that M is closed. Indeed, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(M)$ such that $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y}) \in X \times X$. Then $y_\alpha \in S(x_\alpha)$ and $m(x_\alpha) \in \Phi(x_\alpha, y_\alpha)$. Since Φ is u.s.c. with closed values, it follows from Lemma 1 that Φ is closed. Since S and Φ are closed and m is continuous, we have $\bar{y} \in S(\bar{x})$ and $m(\bar{x}) \in \Phi(\bar{x}, \bar{y})$. Therefore, M is closed. Since S is compact, so is M. Thus M is u.s.c. with compact acyclic values. It follows from Lemma 6 that M has a fixed point $\hat{x} \in X$ such that $\hat{x} \in M(\hat{x})$. That is, $\hat{x} \in S(\hat{x})$ such that $m(\hat{x}) \in \Phi(\hat{x},\hat{x})$. Further, if $\Phi(\hat{x},\hat{x}) \leqslant 0$, then

$$\Phi(\hat{x}, y) \leqslant \sup_{u \in S(\hat{x})} \Phi(\hat{x}, u) = m(\hat{x}) \leqslant 0 \quad \text{for all } y \in S(\hat{x}). \qquad \square$$

Remark 1. For a single-valued function $\Phi = \varphi : X \times X \to \bar{R}$, Theorem 4 reduced to Park [12, Corollary 1], which was applied to variational inequalities, optimization problem, and others in his earlier works.

Remark 2. In Theorem 4, if S and Φ are continuous, then condition (1) is satisfied automatically. This can be seen easily in view of Lemma 5(a).

From Theorem 1, we have another solution of QEPs for a single-valued function $\varphi : X \times X \to \bar{R}$:

Theorem 5. *Let X be a paracompact admissible convex and perfectly normal subset of a t.v.s. E, K a nonempty compact subset of X and $S : X \multimap K$ a map with convex* *values and open fibers such that*

$$\bar{S}: X \multimap K \quad \text{is u.s.c., where } \bar{S}(x) = \overline{S(x)} \text{ for } x \in X.$$

Let $\varphi: X \times X \to \bar{R}$ *be a function such that*
(i) $\varphi(x,z)$ *is u.s.c. in* $x \in X$ *and quasi-convex in* $z \in X$.
(ii) $\varphi(x,x) \geqslant 0$ *for all* $x \in X$.
Then there exists $\hat{x} \in \bar{S}(\hat{x})$ *such that*

$$\varphi(\hat{x},x) \geqslant 0 \quad \text{for all } x \in S(\hat{x}).$$

Proof. For each $x \in X$, let

$$G(x) = \{u \in Sx : \varphi(x,u) < 0\}.$$

Since $u \mapsto \varphi(x,u)$ is quasi-convex and Sx is convex, each Gx is convex. Since $x \mapsto \varphi(x,u)$ is u.s.c. for each $u \in X$, the set $\{x \in X : \varphi(x,u) < 0\}$ is open in X. Moreover, since S has open fibers,

$$G^-(u) = S^-(u) \cap \{x \in X : \varphi(x,u) < 0\}$$

is open in X for each $u \in X$. Let

$$W = \{x \in X : G(x) \neq \emptyset\} = \bigcup_{u \in X} G^-(u).$$

Note that $S: X \multimap K$ has a continuous selection $f: X \to K$ which has a fixed point $\hat{x} = f(\hat{x}) \in K$ by Lemma 6.

Case 1: If $W = \emptyset$, then $G(x) = \emptyset$ for all $x \in X$; that is $\varphi(x,u) \geqslant 0$ for all $x \in X$ and $u \in Sx$. Therefore, $\hat{x} = f(\hat{x}) \in S(\hat{x})$ and $\varphi(\hat{x},x) \geqslant 0$ for all $x \in S(\hat{x})$.

Case 2: If $W \neq \emptyset$, then W is open. Since X is perfectly normal and paracompact, W is paracompact as in [15]. Consequently, by Theorem 1, $G|_W : W \multimap K$ has a continuous selection $g: W \to K$. Define a map $H: (X) \multimap K$ by

$$H(x) = \begin{cases} \{g(x)\} & \text{if } x \in W, \\ \bar{S}(x) & \text{if } x \in X \setminus W. \end{cases}$$

Then by Lemma 3, H is u.s.c. with nonempty closed convex values. Therefore, by Lemma 6, H has a fixed point $\hat{x} \in H(\hat{x})$. If $\hat{x} \in W$, then $\hat{x} = g(\hat{x}) \in G(\hat{x})$. Hence $\varphi(\hat{x},\hat{x}) < 0$ contradicts condition (ii). Therefore, $\hat{x} \in X \setminus W$, which implies $\hat{x} \in \bar{S}(\hat{x})$ and $G\hat{x} = \emptyset$; that is $\varphi(\hat{x},z) \geqslant 0$ for all $z \in S(\hat{x})$.

This completes the proof. □

Remark. If S and φ are continuous, then X is not necessarily perfectly normal; see Remark 2 for Theorem 4.

Corollary 6. *Let* X, K, E *and* S *be the same as in Theorem* 5. *Let* Y *be a nonempty subset of a t.v.s.* F, $T: X \multimap Y$ *a map having a continuous selection* $g: X \to Y$, *and* $\varphi: X \times Y \times X \to \bar{R}$ *a function such that*

(i) $\varphi(x,y,u)$ *is u.s.c. in* (x,y) *and quasi-convex in* u; *and*
(ii) $\varphi(x,y,x) \geqslant 0$ *for each* (x,y) *in the graph of* T.
Then there exists an $\hat{x} \in \bar{S}(\hat{x})$ *and* $\hat{y} \in T(\hat{x})$ *such that*

$$\varphi(\hat{x},\hat{y},x) \geqslant 0 \quad \textit{for all } x \in S(\hat{x}). \qquad \square$$

Proof. Put $\psi(x,z) = \varphi(x,g(x),z)$ and apply Theorem 5.

We give an application of Theorem 5 to a generalized quasi-variational inequality as follows:

Corollary 7. *Let* X *be a paracompact admissible convex and perfectly normal subset of a t.v.s.* E, E^* *its dual space with any topology such that* $\mathrm{Re}\langle , \rangle : E^* \times E \to R$ *is continuous,* K *a nonempty compact subset of* X, *and* $\emptyset \neq Y \subset E^*$. *Let* $S : X \multimap K$ *and* $T : X \multimap Y$ *be maps as in Corollary 6. Then there exist an* $\hat{x} \in \bar{S}(\hat{x})$ *and a* $\hat{y} \in T(\hat{x})$ *such that*

$$\mathrm{Re}\,\langle \hat{y}, \hat{x} - z\rangle \leqslant 0 \quad \textit{for all } z \in S(\hat{x}).$$

Proof. Let $\varphi(x,y,u) = \mathrm{Re}\langle y, x-z\rangle$ in Corollary 6. Then Corollary 7 follows immediately from Corollary 6. $\square$

Remark. Corollary 7 improves [15, Corollary 8].

5. Generalized quasi-equilibrium problems

As we have seen already in the end of Section 4, solutions of certain GQEPs can be derived from that of QEPs. We begin with the following GQEP in this section.

Theorem 8. *Let* X *be an admissible convex subset of a t.v.s.* E, Y *a subset of a t.v.s.,* $S : X \multimap X$ *a compact closed map,* $f : X \to Y$ *a continuous function, and* $\Phi : X \times Y \times X \multimap \bar{R}$ *an u.s.c. map with compact values. Suppose that*
(i) *the function* $m : X \times Y \to \bar{R}$ *defined by*

$$m(x,y) = \sup_{u \in S(x)} \Phi(x,y,u) \quad \textit{for } (x,y) \in X \times Y$$

is l.s.c.; *and*
(ii) *for all* $(x,y) \in X \times Y$.

$$M(x,y) = \{u \in S(x) : m(x,y) \in \Phi(x,y,u)\}$$

is acyclic.
Then there exists an $(\bar{x},\bar{y}) \in X \times Y$ *such that*

$$\bar{x} \in S(\bar{x}),\ \bar{y} = f(\bar{x}) \quad \textit{and} \quad m(\bar{x},\bar{y}) \in \Phi(\bar{x},\bar{y},\bar{x}).$$

Further, if $\Phi(x,y,x) \leqslant 0$ *for all* $x \in X, y \in Y$, *then there exist an* $\bar{x} \in S(\bar{x})$ *and a* $\bar{y} = f(\bar{x})$ *such that*

$$\Phi(\bar{x},\bar{y},x) \leqslant 0 \quad \textit{for all } x \in S(\bar{x}).$$

Proof. Let $h: X \to \bar{R}$ be defined by

$$h(x) = m(x, f(x)) \quad \text{for } x \in X,$$

$\Psi: X \times X \multimap \bar{R}$ by $\Psi(x, y) = \Phi(x, f(x), y)$ for $(x, y) \in X \times X$, and $H: X \multimap X$ by

$$H(x) = M(x, f(x)).$$

By assumption, we see that h is l.s.c., Ψ is u.s.c. with compact values, and $H(x)$ is acyclic for all $x \in X$. Then by Theorem 4, there exists an $\bar{x} \in S(\bar{x})$ such that $h(\bar{x}) \in \Psi(\bar{x}, \bar{x})$; that is, $\bar{y} = f(\bar{x})$ and $m(\bar{x}, \bar{y}) \in \Psi(\bar{x}, \bar{x}) = \Phi(\bar{x}, f(\bar{x}), \bar{x}) = \Phi(\bar{x}, \bar{y}, \bar{x})$. Further, if $\Phi(x, y, x) \leqslant 0$ for all $x \in X, y \in Y$, then

$$\Phi(\bar{x}, \bar{y}, x) \leqslant m(\bar{x}, \bar{y}) \leqslant 0 \quad \text{for all } x \in S(\bar{x}). \qquad \Box$$

Remark. Theorem 8 is a variant of Park [12, Theorem 1], which is not comparable to Theorem 8.

From Theorem 8, we have the following:

Theorem 9. *Let X be a paracompact admissible convex subset of a t.v.s. E, Y a compact subset of a t.v.s. E, $S: X \multimap X$ a compact u.s.c. map with closed values, $T: X \multimap Y$ a map with convex values such that $X = \bigcup_{y \in Y} \operatorname{Int} T^-(y)$, and $\Phi: X \times Y \times X \multimap \bar{R}$ an u.s.c. map with compact values. Suppose that*

(1) *the function $m: X \times Y \to \bar{R}$ defined by*

$$m(x, y) = \sup_{u \in S(x)} \Phi(x, y, u) \quad \textit{for } (x, y) \in X \times Y$$

is l.s.c.; and

(2) *for all $(x, y) \in X \times Y$,*

$$M(x, y) = \{u \in S(x) \colon m(x, y) \in \Phi(x, y, u)\}$$

is acyclic.

Then there exist an $\bar{x} \in S(\bar{x})$ and a $\bar{y} \in T(\bar{x})$ such that

$$m(\bar{x}, \bar{y}) \in \Phi(\bar{x}, \bar{y}, \bar{x}).$$

In particular, if we assume that

$$\Phi(x, y, x) \leqslant 0 \quad \textit{for all } (x, y) \in X \times Y,$$

then

$$\Phi(\bar{x}, \bar{y}, x) \leqslant 0 \quad \textit{for all } x \in S(\bar{x}).$$

Proof. Since X is a paracompact space, it follows from Theorem 1, T has a continuous selection $f: X \to Y$. By Theorem 8, there exist an $\bar{x} \in S(\bar{x})$ and a $\bar{y} = f(\bar{x}) \in T(\bar{x})$ such that

$$m(\bar{x}, \bar{y}) \in \Phi(\bar{x}, \bar{y}, \bar{x}).$$

If $\Phi(x, y, x) \leqslant 0$ for all $(x, y) \in X \times Y$, then there exists a $\bar{z} \in \Phi(\bar{x}, \bar{y}, \bar{x})$ such that

$$m(\bar{x}, \bar{y}) = \bar{z} \leqslant 0.$$

Hence $\Phi(\bar{x}, \bar{y}, x) \leqslant 0$ for all $x \in S(\bar{x})$. $\Box$

Corollary 10. *In Theorem* 9, *if* S *has convex values, then condition* (2) *can be replaced by the following without affecting its conclusion*:
(2)′ *for each* $(x,y)\in X\times Y$, *the multimap* $u \multimap \Phi(x,y,u)$ *is quasi-concave*.

Proof. It suffices to show that (2)′ implies (2). In fact, we show that $M(x,y)$ is convex for each $(x,y)\in X\times Y$. Indeed, if $u_1,u_2\in M(x,y)$ and $\lambda\in[0,1]$, then $u_1,u_2\in S(x)$ and $m(x,y)\in\Phi(x,y,u_1)\cap\Phi(x,y,u_2)$. Since the map $u| \multimap \Phi(x,y,u)$ is quasi-concave, it follows from Lemma 4, there exists $w\in\Phi(x,y,\lambda u_1+(1-\lambda)u_2)$ such that

$$w \geqslant m(x,y).$$

Since $S(x)$ is convex, it follows that $\lambda u_1+(1-\lambda)u_2\in S(x)$. By the definition of $m(x,y)$, we see that

$$m(x,y)=w\in\Phi(x,y,\lambda u_1+(1-\lambda)u_2).$$

This shows $\lambda u_1+(1-\lambda)u_2\in M(x,y)$ and hence $M(x,y)$ is convex for all $(x,y)\in X\times Y$. Now the conclusion follows immediately from Theorem 9. □

Remark. Corollary 10 is a far-reaching sharpened form of Wu and Shen [15, Theorem 4].

Corollary 11. *Let* X *be a paracompact admissible convex subset of a t.v.s.* $E, Y\subset E^*$, *where* E^* *is the topological dual space of* E *with a topology on* E^* *such that* $\langle,\rangle$ *is continuous. Let* K *be a compact subset of* X *and* $S:X\multimap K$ *be a map with closed convex values,* $T:X\multimap Y$ *a map having convex values such that* $X=\bigcup_{y\in Y}\operatorname{Int}T^-(y)$. *Then there exist an* $\bar{x}\in S(\bar{x})$ *and a* $\bar{y}\in T(\bar{x})$ *such that*

$$\operatorname{Re}\langle\bar{y},\bar{x}-z\rangle\leqslant 0 \quad \textit{for all } z\in S(\bar{x}).$$

Proof. Let $\varphi(x,y,z)=\operatorname{Re}\langle y,x-z\rangle$, then $\varphi:X\times Y\times X\to\bar{R}$ is continuous and the map $u\mapsto\varphi(x,y,u)$ is concave. Then the conclusion follows immediately from Theorem 1, Corollary 10 and Lemma 5. □

Theorem 12. *Let* X *be a compact convex subset of a t.v.s.* E *on which its topological dual* E^* *separates points of* E, F *a t.v.s. and* $Y\subset F$.

Let $S:X\multimap X$ *be a continuous map with closed convex values,* $T:X\multimap Y$ *a map having continuous selection* $t:X\to Y$, *and* $\varphi:X\times Y\times X\to\bar{R}$. *Suppose that*
(i) $\varphi(x,y,u)$ *is u.s.c. on* (x,y) *and convex on* u; *and*
(ii) $\varphi(x,y,x)\geqslant 0$ *for each* $(x,y)\in\operatorname{Gr}(T)$.
Then there exist an $\bar{x}\in S(\bar{x})$ *and a* $\bar{y}\in T(\bar{x})$ *such that*

$$\varphi(\bar{x},\bar{y},x)\geqslant 0 \quad \textit{for all } x\in S(\bar{x}).$$

Proof. We will use Lemma 8 to prove this theorem. Note that
(1) S is u.s.c.
(2) Let $f:X\times X\to\bar{R}$ be defined by $f(x,u)=-\varphi(x,t(x),u)$.
Then by (i) for each $u\in X$, $x\mapsto f(x,u)$ is l.s.c.

(3) $f(x,u) = -\varphi(x,t(x),u)$ is concave on u, and hence $f(x,u)$ is 0-DCV in u.
(4) Since $x \mapsto f(x,u)$ is l.s.c. and S is l.s.c., it follows from Lemma 5 that $x \mapsto \sup_{u\in S(x)} f(x,u)$ is l.s.c.. Therefore

$$\left\{x \in X: \sup_{u\in S(x)} f(x,u) \leqslant 0\right\}$$

is closed.

Thus the conclusion follows from Lemma 8. □

Remark. Theorem 12 is a correct and improved version of Wu and Shen [15, Theorem 6].

Final Remark. Wu and Shen [15, Theorem 2] is a collectively fixed point theorem and is already extended by the third author [12] in view of Lemma 6. The extension is applied in [13] to give equilibrium existence theorems for abstract economics. We also note that [15, Theorems 10–12] can be improved following the methods of this paper. Consequently, all the results of Wu and Shen [15] can be generalized and improved.

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