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A PROOF OF THE SPERNER LEMMA FROM THE BROUWER FIXED POINT THEOREM

Sehie Park[†] **and Kwang Sik Jeong**[‡]

[†] *School of Mathematical Sciences*
Seoul National University
Seoul 151–747, Korea
[‡] *Department of Mathematics*
Soongsil University
Seoul 156–743, Korea

Abstract. We give a proof of the Sperner lemma from the Brouwer fixed point theorem. Our proof is much simpler than the existing one.

It is well-known that the Brouwer fixed point theorem has numerous equivalent formulations and applications in various fields of mathematics such as topology, nonlinear analysis, equilibrium theory in economics, game theory, and others.

In 1912, the following Brouwer fixed point theorem appeared in [4]:

The Brouwer Theorem. *A continuous map* $f : \Delta \to \Delta$ *has a fixed point* $x_0 = f(x_0) \in \Delta$.

Here, Δ denotes an n-simplex with vertices $v_0, v_1, \cdots, v_n$ and $\partial\Delta$ the boundary of Δ. Let a face $v_{i_0}v_{i_1}\cdots v_{i_k}$ of Δ be the set $\{\sum_{j=0}^{k}\lambda_j v_{i_j} | \lambda_j > 0 \text{ and } \sum_{j=0}^{k}\lambda_j = 1\}$ and $\overline{v_{i_0}v_{i_1}\cdots v_{i_k}} = \{\sum_{j=0}^{k}\lambda_j v_{i_j} | \lambda_j \geq 0 \text{ and } \sum_{j=0}^{k}\lambda_j = 1\}$ be the closure of the face $v_{i_0}v_{i_1}\cdots v_{i_k}$, where $0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n$. For convenience, v_{n+1} denotes v_0 if necessary.

In 1928, Sperner [12] obtained a purely combinatorial lemma concerned with the nature of certain labellings of the vertices of subdivisions of simplexes:

The Sperner Lemma. *Let* K *be a simplicial subdivision of an* n*-simplex* $\Delta = \overline{v_0v_1\cdots v_n}$. *To each vertex* u *of* K, *let an integer* $s(u)$ *be assigned in such a way that whenever* u *lies on a face* $v_{i_0}v_{i_1}\cdots v_{i_k}$, *the number* $s(u)$ *assigned to* u *is one of the integers* $i_0, i_1, \cdots, i_k$. *Then there is an* n*-simplex of* K, *whose vertices receive all* $n+1$ *integers* $0, 1, \cdots, n$.

For proofs of the Sperner lemma, see [1, 2, 3, 6, 7, 12, 15]. The lemma was first applied to new proofs of the invariance theorems on dimensions and domains in [12], and subsequently, to obtain the following due to Knaster, Kuratowski and Mazurkiewicz (simply, KKM) [8]:

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The KKM Theorem. *Let F_i $(0 \le i \le n)$ be $n+1$ closed subsets of an n-simplex $\overline{v_0 v_1 \cdots v_n}$. If the inclusion relation*

$$v_{i_0} v_{i_1} \cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

holds for all faces $v_{i_0} v_{i_1} \cdots v_{i_k}$, then $\bigcap_{i=0}^{n} F_i \neq \emptyset$.

For proofs using the Sperner lemma, see [1, 2, 3, 6, 7, 12, 15]. The KKM theorem was used in [8] to obtain one of the most direct proofs of the Brouwer theorem. Therefore, it was conjectured that those three theorems are mutually equivalent. This was clarified by Yoseloff [13] in 1974. In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications.

Brouwer (1912)

1974 $\swarrow$ $\nwarrow$ 1929

Sperner (1928) $\xrightarrow[1929]{}$ KKM (1929)

There are also proofs of the KKM theorem using the Brouwer theorem; for example, see [5, 7, 14]. However, Yoseloff's one seems to be the only proof of the Sperner lemma using the Brouwer theorem. This might be a standard exercise.(See Exercise 10 of Chapter 14 of Zangwill and Garcia [14].)

In this short note, we deduce the Sperner lemma from the Brouwer theorem with much shorter proof than Yoseloff's.

Proof of the Sperner lemma using the Brouwer theorem. Suppose that there is no n-simplex of K whose vertices receive all integers $0, 1, \cdots, n$. Let K^0 and Δ^0 be the set of all vertices of K and Δ respectively. Define a map $f : K^0 \to \Delta^0$ by $f(u) = v_{s(u)+1}$ for all $u \in K^0$. We extend f to $F : K(= \Delta) \to \Delta$ by $F(x) = \sum_{j=0}^{n} \lambda_j f(u_j)$ for each $x = \sum_{j=0}^{n} \lambda_j u_j$ in some n-simplex $\overline{u_0 u_1 \cdots u_n}$ of K.

Then F is well defined and continuous. Since all $f(u_0), f(u_1), \cdots, f(u_n)$ are not different, $F(x)$ lies on $\partial\Delta$ for all $x \in \Delta$.

Moreover, if $x \in \partial\Delta$, then there exist a face $u_0 u_1 \cdots u_k$ of K and a face $v_{i_0} v_{i_1} \cdots v_{i_m}$, $0 \le k \le m < n$, of Δ such that $x \in u_0 u_1 \cdots u_k \subset v_{i_0} v_{i_1} \cdots v_{i_m}$. Let $x = \sum_{j=0}^{k} \lambda_j u_j$. Then

$$F(x) = \sum_{j=0}^{k} \lambda_j f(u_j) = \sum_{j=0}^{k} \lambda_j v_{s(u_j)+1} \in \overline{v_{i_0+1} v_{i_1+1} \cdots v_{i_m+1}}$$

since $s(u_j) \in \{i_0, i_1, \cdots i_m\}$ for all $j = 0, \cdots, k$.

Note that $v_{i_0} v_{i_1} \cdots v_{i_m} \cap \overline{v_{i_0+1} v_{i_1+1} \cdots v_{i_m+1}} = \emptyset$. Hence F has no fixed point in $\partial\Delta$. Therefore F has no fixed point in Δ. This contradicts the Brouwer theorem.

Finally, for the history or equivalent formulations of the Brouwer theorem and related topics, see [9, 10, 11].

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