

# FIXED POINTS OF CONVEX-VALUED GENERALIZED UPPER HEMICONTINUOUS MAPS, REVISITED

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ABSTRACT. We give new fixed point theorems on a generalized upper hemicontinuous multimap whose domain and range may have different topologies. These include known theorems appeared in almost 50 published works.

## 1. Introduction

The celebrated Kakutani fixed point theorem in 1941 for convex-valued upper semicontinuous multimaps initiated the study of fixed points of multimaps in the last six decades. The Kakutani theorem and its numerous generalizations were applied to game theory, mathematical economics, systems and control theory, coincidence theory, minimax theory, variational inequalities, convex analysis, and many equilibrium theorems. Moreover, the compactness, convexity, upper semicontinuity, selfmapness, and finite dimensionality related to the Kakutani theorem are all extended, and further, for the case of infinite dimension, it is known that the domain and range of the multimap may have different topologies. This is why the Kakutani theorem has so many generalizations; see [P7].

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In our previous works [P5,6], we unified, improved, and generalized a lot of fixed point theorems on Kakutani maps or acyclic maps defined on convex subsets of topological vector spaces. One of the main fixed point theorems in [P6] is concerned with convex-valued generalized upper hemicontinuous maps whose domains and ranges may have different topologies. After the author published the paper [P6], he became aware that there still appear a number of results of this kind of generalization.

In the present paper, we obtain some refined and generalized versions of main theorems in [P5,6] with slightly different proofs. We also show that some old or recent results of others are consequences of ours. Our results contain known theorems of Sehgal and Singh [SS], Roux and Singh [RS], Kim and Tan [KT], Ding and Tan [DT], Yuan, Smith, and Lou [YSL], and many others.

## 2. Preliminaries

A *convex space*  $X$  is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset  $L$  of a convex space  $X$  is called a *c-compact set* if for each finite set  $S \subset X$  there is a compact convex set  $L_S \subset X$  such that  $L \cup S \subset L_S$ . Let  $[x, L]$  denote the closed convex hull of  $\{x\} \cup L$  in  $X$ , where  $x \in X$ .

Let  $E$  be a Hausdorff topological vector space (t.v.s.) and  $E^*$  its topological dual. A multimap or set-valued map (simply, map)  $F : X \rightarrow 2^E \setminus \{\emptyset\}$  is said to be *upper hemicontinuous* (u.h.c.) if for each  $h \in E^*$  and for any real  $\alpha$ , the set  $\{x \in X : \sup \operatorname{Re} h(Fx) < \alpha\}$  is open in  $X$ .

Let  $cc(E)$  denote the set of nonempty closed convex subsets of  $E$  and  $kc(E)$  the set of nonempty compact convex subsets of  $E$ .  $\operatorname{Bd}$ ,  $\operatorname{Int}$ , and  $\overline{\phantom{x}}$  denote the boundary, interior, and closure, resp., with respect to  $E$ .

Let  $X \subset E$  and  $x \in E$ . The *inward* and *outward sets* of  $X$  at  $x$ ,  $I_X(x)$  and  $O_X(x)$ , are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

For  $p \in \{\operatorname{Re} h : h \in E^*\}$  and  $U, V \subset E$ , let

$$d_p(U, V) = \inf\{|p(u - v)| : u \in U, v \in V\}.$$

Recall that a real function  $g : X \rightarrow \mathbb{R}$  on a topological space  $X$  is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if  $\{x \in X : gx > r\}$  [resp.  $\{x \in X : gx < r\}$ ] is open for each  $r \in \mathbb{R}$ . If  $X$  is a convex set, then  $g$  is *quasiconcave* [resp. *quasiconvex*] if  $\{x \in X : gx > r\}$  [resp.  $\{x \in X : gx < r\}$ ] is convex for each  $r \in \mathbb{R}$ .

In this paper all topological spaces are assumed to be Hausdorff.

We use the following form of the existence theorem of maximizable quasiconcave functions on convex spaces due to Park and Bae [PB].

**Theorem 0.** *Let  $X$  be a convex space and  $\hat{X}$  the set of all u.s.c. quasiconcave real functions on  $X$ . Suppose that*

- (0.1) *for each  $x \in X$ ,  $Sx$  is a nonempty convex subset of  $\hat{X}$ ;*
- (0.2) *for each  $g \in \hat{X}$ ,  $S^{-1}g$  is compactly open in  $X$ ; and*
- (0.3) *there exists a  $c$ -compact set  $L \subset X$  and a nonempty compact set  $K \subset X$  such that for every  $x \in X \setminus K$  and  $g \in Sx$ ,  $gx < \max g[x, L]$ .*

*Then there exist an  $\bar{x} \in K$  and a  $g \in S\bar{x}$  such that  $g\bar{x} = \max g(X)$ .*

### 3. Main results

We begin with the following generalization of [P5, Corollary 3.1]:

**Theorem 1.** *Let  $X$  be a convex space,  $L$  a  $c$ -compact subset of  $X$ ,  $K$  a nonempty compact subset of  $X$ ,  $E$  a t.v.s. containing  $X$  as a subset, and  $F : X \rightarrow 2^E \setminus \{\emptyset\}$ . Suppose that, for each  $p \in \{\operatorname{Re} h : h \in E^*\}$ ,*

- (1.0)  *$p|_X$  is continuous on  $X$ ;*
- (1.1)  *$X_p = \{x \in X : \sup p(Fx) \geq p(x)\}$  is compactly closed in  $X$ ;*
- (1.2)  *$x \in K$  and  $p(x) = \max p(X)$  implies  $x \in X_p$ ; and*
- (1.3)  *$x \in X \setminus K$  and  $p(x) = \max p[x, L]$  implies  $x \in X_p$ .*

Then there exists an  $x \in \bigcap \{X_p : p \in \{\operatorname{Re} h : h \in E^*\}\}$ .

*Proof.* Note that  $\{(\operatorname{Re} h)|_X : h \in E^*\} \subset \hat{X}$  by (1.0). For each  $x \in X$ , define

$$Sx = \{p|_X : p \in \{\operatorname{Re} h : h \in E^*\} \text{ and } \sup p(Fx) < p(x)\}.$$

Then  $Sx$  is a convex subset of  $\hat{X}$ . Suppose that  $Sx \neq \emptyset$  for each  $x \in X$ ; that is, for each  $x \in X$ , there exists a  $p \in \{\operatorname{Re} h : h \in E^*\}$  such that  $x \notin X_p$ . Note that, for each  $g \in \hat{X}$ ,

$$S^{-1}g = \{x \in X : \sup p(Fx) < p(x)\} = X \setminus X_p$$

if  $g = p|_X$  for some  $p \in \{\operatorname{Re} h : h \in E^*\}$  and

$$S^{-1}g = \emptyset \quad \text{if } g \notin \{(\operatorname{Re} h)|_X : h \in E^*\}.$$

Then  $S^{-1}g$  is compactly open in  $X$  for each  $g \in \hat{X}$  by (1.1). Therefore, (0.1) and (0.2) are satisfied. Further, (1.3) implies (0.3). In fact, for every  $x \in X \setminus K$  and  $p \in \{\operatorname{Re} h : h \in E^*\}$  satisfying  $\sup p(Fx) < p(x)$ , we have  $x \notin X_p$ . Therefore,  $p(x) < \max p[x, L]$  by (1.3). Now, by applying Theorem 0, there exist an  $\bar{x} \in K$  and an  $h \in E^*$  such that  $p = \operatorname{Re} h$ ,  $p|_X \in S\bar{x}$  and  $p(\bar{x}) = \max p(X)$ . Note that  $p|_X \in S\bar{x}$  implies  $\bar{x} \notin X_p$ . This contradicts (1.2).

**Remarks.** 1. As we noted in [P6], in Theorem 1, we do not require any concrete connection between topologies of  $X$  and  $E$  except

$$(1.0) \quad (\operatorname{Re} h)|_X \in \hat{X} \quad (\text{that is, } (\operatorname{Re} h)|_X \text{ is continuous on } X) \quad \text{for all } h \in E^*.$$

In order to assure the continuity of  $(\operatorname{Re} h)|_X$  for all  $h \in E^*$ , it is sufficient to assume that

(i) as a convex space,  $X$  has any topology finer than the relative weak topology with respect to  $E$ , and

(ii)  $E$  has any topology finer than its weak topology.

This is why there have appeared fixed point theorems on maps whose domains and ranges have different topologies.

2. If  $F$  is u.h.c. on each nonempty compact subset  $C$  of  $X$ , then  $F$  satisfies the “continuity” condition (1.1) for all  $p \in \{\text{Re } h : h \in E^*\}$ , but not conversely; see [P5]. Any map  $F$  satisfying (1.1) can be said to be *generalized u.h.c.*

3. The “boundary” condition (1.2) is equivalent to the following:

$$(1.2)' \quad x \in K \text{ and } p(x) = \max p(\bar{I}_X(x)) \text{ implies } x \in X_p.$$

In fact,  $p(x) = \max p(X)$  is equivalent to  $p(x) = \max p(\bar{I}_X(x))$ .

Let  $X$  be a nonempty convex subset of a vector space  $E$ . Following Fan [F], the *algebraic boundary*  $\delta_E(X)$  of  $X$  in  $E$  is the set of all  $x \in X$  for which there exists  $y \in E$  such that  $x + ry \notin X$  for all  $r > 0$ . If  $E$  is a t.v.s., the *topological boundary*  $\text{Bd } X = \text{Bd}_E X$  of  $X$  is the complement of  $\text{Int}_E X$  in  $\bar{X}$ . It is known that  $\delta_E(X) \subset \text{Bd } X$  and in general  $\delta_E(X) \neq \text{Bd } X$ ; see [YSL].

Moreover, the “boundary” condition (1.2)' is equivalent to the following:

$$(1.2)'' \quad x \in K \cap \delta_E(X) \text{ and } p(x) = \max p(\bar{I}_X(x)) \text{ implies } x \in X_p.$$

In fact, if  $x \in K \setminus \delta_E(X)$  and  $p(x) = \max p(\bar{I}_X(x))$ , then for any  $y \in E$ , there exists an  $r > 0$  such that  $x + ry \in X$  and hence  $p(x) \geq p(x + ry)$ , which readily implies  $p(y) \leq 0$  or  $p = 0$ . This contradicts the arbitrariness of  $p$ . Therefore, (1.2) is trivially satisfied.

4. The “coercivity” or “compactness” condition (1.3) is equivalent to the following:

$$(1.3)' \quad x \in X \setminus K \text{ and } p(x) = \max p(\bar{I}_L(x)) \text{ implies } x \in X_p.$$

In fact,  $p(x) = \max p[x, L]$  is equivalent to  $p(x) = \max p(\bar{I}_L(x))$ . Note that if  $X$  itself is compact (that is, if  $X = K$ ), then (1.3)' holds trivially.

From Theorem 1, we have the following basic fixed point theorem:

**Theorem 2.** *Under the hypothesis of Theorem 1, further suppose that either*

(A)  *$E^*$  separates points of  $E$  and  $F : X \rightarrow kc(E)$ ; or*

(B)  *$E$  is locally convex and  $F : X \rightarrow cc(E)$ .*

*Then there exists an  $x \in X$  such that  $x \in Fx$ .*

*Proof.* By Theorem 1, there exists an  $x \in \bigcap \{X_p : p \in \{\text{Re } h : h \in E^*\}\}$ . Suppose that  $x \notin Fx$ . Then under the assumptions (A) or (B), the standard separation theorems on a t.v.s. assure the existence of a  $p \in \{\text{Re } h : h \in E^*\}$  satisfying  $\inf p(Fx) > p(x)$ ; that is,  $x \notin X_p$ , which is a contradiction.

**Remarks.** 1. Using the method in [P5], we can reformulate Theorem 2 to a coincidence theorem and an existence theorem for critical points or zeros of multimaps.

2. Note that  $x \in Fx$  if and only if  $x \in \bigcap \{X_p : p \in \{\text{Re } h : h \in E^*\}\}$ . This is a useful information on the location of a fixed point.

From Theorem 2, we obtain the following more visualizable geometric form of a fixed point and surjectivity theorem, which generalizes [P5, Theorem 6] and refines [P6, Theorem 2]:

**Theorem 3.** *Let  $X$  be a convex space,  $L$  a  $c$ -compact subset of  $X$ ,  $K$  a nonempty compact subset of  $X$ ,  $E$  a t.v.s. containing  $X$  as a subset, and  $F$  a map satisfying either*

(A)  *$E^*$  separates points of  $E$  and  $F : X \rightarrow kc(E)$ , or*

(B)  *$E$  is locally convex and  $F : X \rightarrow cc(E)$ .*

(I) *Suppose that for each  $p \in \{\text{Re } h : h \in E^*\}$ ,*

(1.0)  *$p|_X$  is continuous on  $X$ ;*

(3.1)  *$X_p = \{x \in X : \inf p(Fx) \leq p(x)\}$  is compactly closed in  $X$ ;*

(3.2)  *$d_p(Fx, \bar{I}_X(x)) = 0$  for every  $x \in K \cap \delta_E(X)$ ; and*

(3.3)  *$d_p(Fx, \bar{I}_L(x)) = 0$  for every  $x \in X \setminus K$ .*

Then there exists an  $x \in X$  such that  $x \in Fx$ .

(II) Suppose that for each  $p \in \{\operatorname{Re} h : h \in E^*\}$ ,

(1.0)  $p|_X$  is continuous on  $X$ ;

(3.1)'  $X_p = \{x \in X : \sup p(Fx) \geq p(x)\}$  is compactly closed in  $X$ ;

(3.2)'  $d_p(Fx, \overline{O}_X(x)) = 0$  for every  $x \in K \cap \delta_E(X)$ ; and

(3.3)'  $d_p(Fx, \overline{O}_L(x)) = 0$  for every  $x \in X \setminus K$ .

Then there exists an  $x \in X$  such that  $x \in Fx$ . Further, if  $F$  is u.h.c., then  $F(X) \supset X$ .

*Proof.* In order to use Theorem 2, we first show that (3.2)  $\implies$  (1.2). Let  $x \in K \cap \delta_E(X)$  such that  $p(x) = \max p(X)$ . Suppose that  $\inf p(Fx) > p(x)$ . Then for any  $v \in Fx$ ,  $u \in X$ ,  $z = x + r(u - x) \in I_X(x)$ , and  $r > 0$ , we have

$$|p(v - z)| = p(v - x) + rp(x - u) \geq p(v - x) = p(v) - p(x)$$

and hence

$$d_p(Fx, \overline{I}_X(x)) = d_p(Fx, I_X(x)) \geq \inf p(Fx) - p(x) > 0.$$

This contradicts (3.2). Therefore, we should have  $\inf p(Fx) \leq p(x)$  or  $x \in X_p$ . Hence, (1.2)'' holds.

Similarly, we can show that (3.3)  $\implies$  (1.3). Note that “(3.1) holds for all  $p$ ” is equivalent to “(1.1) = (3.1)' holds for all  $p$ ”. Therefore, all of the requirements of Theorem 2 are satisfied. Now by Theorem 2, Case (I) follows.

For (II) consider  $2x - Fx$  instead of  $Fx$  in (I) as in [P5], we can conclude that  $F$  has a fixed point. For the surjectivity result, let  $y \in X$ . Consider  $x \mapsto Fx + x - y$  instead of  $Fx$  and  $[y, L]$  instead of  $L$  in Case (II). Then there exists an  $x \in X$  such that  $x \in Fx + x - y$ ; that is,  $y \in Fx$ . This completes our proof.

**Remarks.** 1. (3.1) and (3.1)' are actually the same.

2. Note that the map  $x \mapsto Fx + x - y$  in the proof of Case (II) is u.h.c.

3. Note that if  $K$  is a weakly compact convex subset of a t.v.s.  $(E, \tau)$  on which  $E^*$  separates points, then a continuous map  $f : (K, \tau) \rightarrow (K, \tau)$  may have no fixed point. See Kakutani [K, Theorem 1]. In this case,  $K_p = \{x \in K : p(fx) \leq p(x)\}$  in Theorem 3(I) may not be closed for some  $p \in \{\text{Re } h : h \in E^*\}$ .

We give some of the simplest examples of Theorem 3.

**Examples.** 1. [P2, Example 1]: Let  $X = K = [0, 1]$  in  $E = \mathbb{R}$ ,  $fx = x$  for  $x \in X \setminus (1/3, 2/3)$ , and  $fx = 1$  for  $x \in (1/3, 2/3)$ . Then the set  $\{x \in X : p(x) \leq p(fx)\}$  is closed for all  $p \in E^*$ . Note that  $f : X \rightarrow X$  is not continuous, but has a fixed point by Theorem 3(I).

2. [P2, Example 2]: Let  $X = K = [0, 1]$  and  $E = \mathbb{R}$ . For a given  $c \in (0, 1)$ , let  $f : X \rightarrow X$  be a function such that  $fx > x$  for  $x < c$  and  $fx < x$  for  $x > c$ . Then  $c$  is the only fixed point of  $f$  if and only if the set  $\{x \in X : p(x) \leq p(fx)\}$  is closed for all  $p \in E^*$ .

3. Let  $X = (0, 1]$ ,  $K = L = [1/2, 1]$ ,  $E = \mathbb{R}$ , and  $f : X \rightarrow X$  be given by  $fx = (x + 1)/2$ . Then  $\{x \in X : p(x) \leq p(fx)\}$  is closed for all  $p \in E^*$ . Note that

$$fx = \frac{x + 1}{2} \in \bar{I}_L(x) = [x, \infty) \quad \text{for all } x \in (0, \frac{1}{2}) = X \setminus K.$$

Therefore, Theorem 3(I) works.

#### 4. Particular results

(1) A particular form of Theorem 3 for the real case is given as [P5, Theorem 6], which unifies, improves, and generalizes historically well-known fixed point theorems published in nearly 40 papers; see the diagram in [P5, p.205].

Now we add some more known consequences of Theorem 3 as follows:



(2) Knaster, Kuratowski, and Mazurkiewicz [KKM, p.136]: If  $f : B^n \rightarrow \mathbb{R}^n$  is a continuous map such that  $f$  maps  $S^{n-1} = \text{Bd } B^n$  back into  $B^n$ , then  $f$  has a fixed point.

This is the origin of the so-called Rothe boundary condition.

(3) Sehgal and Singh [SS, Corollary 2]: Let  $K$  be a convex and weakly compact subset of a real locally convex t.v.s.  $E$  and  $f : K \rightarrow E$  a strongly continuous map such that  $f(\text{Bd } K) \subset K$ . Then  $f$  has a fixed point.

Note that  $f$  satisfies (3.1).

(4) Deimling [D, p.93]: Let  $X$  be a nonempty closed bounded convex subset of a reflexive Banach space  $E$ , and  $f : X \rightarrow X$  a weakly sequentially continuous map. Then  $f$  has a fixed point.

Equip  $E$  with the weak topology.

(5) Arino, Gautier, and Penot [AGP, Theorem 1]: Let  $X$  be a nonempty weakly compact convex subset of a metrizable locally convex t.v.s.  $E$ , and  $f : X \rightarrow X$  a weakly sequentially continuous. Then  $f$  has a fixed point.

Note that  $f$  is weakly continuous.

(6) Roux and Singh [RS, Theorem 5]: Let  $(E, \tau)$  be a t.v.s. on which  $E^*$  separates points,  $w$  the weak topology of  $E$ ,  $K$  a nonempty  $\tau$ -compact convex subset of  $E$ , and  $f : (K, \tau) \rightarrow (E, w)$  a continuous inward map. Then  $f$  has a fixed point.

Here, inward means  $fx \in I_K(x)$  for all  $x \in K$ .

(7) Roux and Singh [RS, Theorem 6]: Let  $(E, \tau)$  be a t.v.s. on which  $E^*$  separates points,  $w$  the weak topology of  $E$ ,  $K$  a nonempty  $w$ -compact convex subset of  $E$ , and  $f : (K, w) \rightarrow (E, \tau)$  a continuous inward map. Then  $f$  has a fixed point.

This contains some results in Sehgal, Singh, and Whitfield [SSW].

(8) Park [P2, Theorem]: Let  $X$  be a nonempty compact convex subset of a t.v.s.  $E$  on which  $E^*$  separates points, and  $f : X \rightarrow E$  a weakly inward [outward] map such that

$$\{x \in X : \text{Re } h(x) < \text{Re } h(fx)\}$$

is open for all  $h \in E^*$ . Then  $f$  has a fixed point.

Here, weakly inward means  $fx \in \bar{I}_X(x)$  for all  $x \in X$ .

(9) Kim and Tan [KT, Theorem 2]: Let  $X$  be a nonempty paracompact bounded convex subset of a locally convex t.v.s.  $E$ ,  $K$  a nonempty compact subset of  $X$ , and  $F : X \rightarrow cc(E)$  an u.h.c. map satisfying the following:

- (a) for each  $x \in X$ ,  $Fx \cap \bar{I}_X(x) \neq \emptyset$ ; and
- (b) for each  $x \in X \setminus K$ ,  $y \in X$  and  $h \in E^*$ , if  $\operatorname{Re} h(y) > \inf \operatorname{Re} h(Fy)$ , then  $\operatorname{Re} h(y) \leq \operatorname{Re} h(x)$ .

Then there exists an  $\hat{x} \in X$  such that  $\hat{x} \in F\hat{x}$ .

Choose a point  $y \in X$  and let  $L = \{y\}$ . If we replace (b) by

- (b)' for each  $x \in X \setminus K$  and  $h \in E^*$ ,  $\operatorname{Re} h(x) < \inf \operatorname{Re} h(Fx)$  implies  $\operatorname{Re} h(y) > \operatorname{Re} h(x)$ .

Then (b)' implies (1.3), which is equivalent to (3.3). See Jiang [J]. Therefore, in this case, the result follows from Theorem 3(B) for Case (I).

Actually, Kim and Tan based their argument on [KT, Corollary 2], which is quite different from our results.

(10) Kim and Tan [KT, Theorem 4]: Let  $X$  be a nonempty convex subset of a normed vector space  $E$ ,  $K$  a nonempty compact subset of  $X$ , and  $F : X \rightarrow cc(E)$  an u.h.c. map satisfying (a) and (b) in (9). Then there exists an  $\hat{x} \in X$  such that  $\hat{x} \in F\hat{x}$ .

In this result, if we replace (b) by (b)', then it also follows from Theorem 3(B) for Case (I). Further, note that in a normed vector space  $E$ , for any  $A, B \in cc(E)$ ,  $d(A, B) = 0$  if and only if  $d_p(A, B) = 0$  for all  $p \in \{\operatorname{Re} h : h \in E^*\}$ , where  $d$  denotes the induced metric. See Jiang [J].

(11) Ding and Tan [DT2, Corollary 3]: Let  $X$  be a nonempty convex subset of a normed vector space  $E$ , and  $G : X \rightarrow kc(E)$  continuous on each nonempty compact subset  $C$  of  $X$ . Suppose that there exist a nonempty compact convex subset  $L$  of  $X$  and a nonempty compact subset  $K$  of  $X$  such that

- (i) for each  $y \in K$ ,  $Gy \cap \bar{I}_X(y) \neq \emptyset$  [resp.  $Gy \cap \bar{O}_X(y) \neq \emptyset$ ];
- (ii) for each  $y \in X \setminus K$ ,  $Gy \cap \bar{I}_L(y) \neq \emptyset$  [resp.  $Gy \cap \bar{O}_L(y) \neq \emptyset$ ].

Then  $G$  has a fixed point.

This result contains Browder [Br, Corollaries 2 and 2'] and Shih and Tan [ST, Corollary 1].

(12) Yuan, Smith, and Lou [YSL]: Using Theorem 0 due to Park and Bae [PB], they proved some coincidence theorems for u.h.c. multimaps in t.v.s., and, as applications, coincidence theorems and several matching theorems for closed coverings of convex sets were derived.

Most of results in [YSL] are consequences or slight variations of earlier works of Park [P3-6]. Interested readers may compare these results. Especially, Theorems 3, 3' and Corollaries 4, 4', 6, 6' in [YSL] are consequences of Theorem 3 of this paper.

**Final Remark.** The major particular forms of Theorem 3 can be adequately summarized by the following enlarged version of the diagrams given in [P1,4,5]. For the references which are not appeared in the end of this paper, see [P5,7].

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex Hausdorff topological vector spaces, and IV for topological vector spaces having sufficiently many linear functionals. Moreover,  $f$  stands for single-valued maps and  $F$  for set-valued maps; and  $K$  stands for a nonempty compact convex subset of a space  $E$ , and  $X$  for a nonempty convex subset of  $E$  satisfying certain coercivity conditions with respect to  $F : X \rightarrow 2^E$  with certain boundary conditions.

In fact, Theorem 3 contains all of the fixed point theorems in the diagram. Note that, in the diagram, Bohl's theorem [Bo] in 1904 was well-known to be equivalent to Brouwer's theorem in 1912.

$E$	$f : K \longrightarrow K$	$F : K \longrightarrow 2^K$
I	Brouwer 1912	Kakutani 1941
II	Schauder 1927, 1930	Bohnenblust and Karlin 1950
III	Tychonoff 1935	Fan 1952 Glicksberg 1952
IV	Fan 1964	Granas and Liu 1986
	$f : K \longrightarrow E$	$F : K \longrightarrow 2^E$
I	Bohl 1904 Knaster, Kuratowski and Mazurkiewicz 1929	
II	Rothe 1938	
III	Halpern 1965 Fan 1969 Reich 1972 Sehgal and Singh 1983	Browder 1968 Fan 1969 Glebov 1969 Halpern 1970 Cellina 1970 Reich 1972, 1978 Cornet 1975 Lasry and Robert 1975 Simons 1986
IV	Halpern and Bergman 1968 Kaczynski 1983 Roux and Singh 1989 Sehgal, Singh and Whitfield 1990	Granas and Liu 1986 Park 1988, 1991
		$F : X \longrightarrow 2^E$
II		Ding and Tan 1992
III		Fan 1984 Shih and Tan 1987, 1988 Jiang 1988
IV		Park 1992, 1993 Yuan, Smith, and Lou 1998

## REFERENCES

- [AGP] O. Arino, S. Gautier, and J.P. Penot, *A fixed point theorem for sequentially continuous mappings with applications to ordinary differential equations*, Funk. Ekv. **27** (1984), 273–279.
- [Bo] P. Bohl, *Über die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage*, J. Reine Angew. Math. **127** (1904), 179–276.
- [Br] F.E. Browder, *On a sharpened form of the Schauder fixed point theorems*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 4749–4751.
- [D] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
- [DT1] X.P. Ding and K.-K. Tan, *A set-valued generalization of Fan's best approximation theorem*, Canadian J. Math. **44** (1992), 784–796.
- [DT2] ———, *A minimax inequality with applications to existence of equilibrium point and fixed point theorems*, Colloq. Math. **63** (1992), 233–247.
- [F] Ky Fan, *Extentions of two fixed point theorems of F. E. Browder*, Math. Z. **112** (1969), 234–240.
- [J] J. Jiang, *Fixed point theorems for paracompact sets*, Acta Math. Sinica **4** (1988), 64–71.
- [K] S. Kakutani, *Topological properties of the unit sphere of a Hilbert space*, Proc. Imp. Acad. Tokyo **19** (1943), 269–271.
- [KT] W.K. Kim and K.-K. Tan, *A variational inequality in non-compact sets and its applications*, Bull. Austral. Math. Soc. **46** (1992), 139–148.
- [KKM] B. Knaster, C. Kuratowski und S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für  $n$ -dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.
- [P1] Sehie Park, *Fixed point theorems on compact convex sets in topological vector spaces*, Contemp. Math. Amer. Math. Soc. **72** (1988), 183–191.
- [P2] ———, *A generalization of the Brouwer fixed point theorem*, Bull. Korean Math. Soc. **28** (1991), 33–37.
- [P3] ———, *Generalized matching theorems for closed coverings of convex sets*, Numer. Funct. Anal. and Optimiz. **11** (1990), 101–110.
- [P4] ———, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, Ed.), World Scientific Publ., River Edge, NJ, 1992, pp.248–277.
- [P5] ———, *Fixed point theory of multifunctions in topological vector spaces*, J. Korean Math. Soc. **29** (1992), 191–208.
- [P6] ———, *Fixed point theory of multifunctions in topological vector spaces, II*, J. Korean Math. Soc., **30** (1993), 413–431.
- [P7] ———, *Eighty years of the Brouwer fixed point theorem*, Antipodal Points and Fixed Points (by J. Jaworowski, W.A. Kirk, and S. Park), Lect. Notes Ser. 28, RIM-GARC, Seoul Nat. Univ., 1995, pp. 55–97.
- [PB] Sehie Park and J.S. Bae, *Existence of maximizable quasiconcave functions on convex spaces*, J. Korean Math. Soc. **28** (1991), 285–292.
- [RS] D. Roux and S.P. Singh, *On a best approximation theorem*, Jñānābha **19** (1989), 1–9.
- [S] J. Schauder, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Z. **26** (1927), 47–65.
- [SS] V.M. Sehgal and S.P. Singh, *A variant of a fixed point theorem of Ky Fan*, Indian J. Math. **25** (1983), 171–174.
- [SSW] V.M. Sehgal, S.P. Singh, and J.H.M. Whitfield, *KKM-maps and fixed point theorems*, Indian J. Math. **32** (1990), 289–296.

- [ST] M.H. Shih and K.-K. Tan, *A geometric property of convex sets with applications to minimax type inequalities and fixed point theorems*, J. Austral. Math. Soc. **45** (1988), 169–183.
- [YSL] G.X.-Z. Yuan, B. Smith, and S. Lou, *Fixed point and coincidence theorems of set-valued mappings in topological vector spaces with some applications*, Nonlinear Analysis, TMA **32** (1998), 183–199.