EXTENSIONS OF MONOTONE SETS*

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Far-reaching generalizations of the extension theorem of monotone sets due to Debrunner and Flor are obtained. In fact, Browder's extension theorem involving Kakutani multimaps is extended to the one involving a large class of "better" admissible maps. Moreover, a sharpened version of the extension theorem for noncompact case due to Lassonde is also obtained.

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1. Introduction

In 1964, Debrunner and Flor [DB] proved an extension theorem of monotone sets. This generalized earlier works of Minty [M1] and Grünbaum [G], which have interesting applications to nonlinear elliptic boundary value problems (see [B1]) and monotone operator theory (for example, see [B2], [Z]). Further generalized and sharpened forms of the extension theorem were obtained by Fan [F], Browder [B3], Lassonde [L], and others.

Especially, Browder [B3] obtained an extension theorem of monotone sets involving the Kakutani multimaps. In the present paper, we extend this result to the one involving a large class of multimaps which were called "better" admissible by the present author [P2]. Moreover, we sharpen the noncompact version of the extension theorem due to Lassonde [L].

2. Preliminaries

In this paper, t.v.s. means topological vector spaces.

Given two t.v.s. E and F, let $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ be a bilinear pairing which is continuous on compact subsets of $F \times E$. This assumption is quite natural in most applications, since the natural pairing between a Hausdorff locally convex t.v.s. E and its dual space E^* equipped with the strong topology enjoys this property.

A subset $M \subset E \times F$ is said to be *monotone* if for any two points (u, w) and (u', w') in M, we have $\langle w - w', u - u' \rangle \geq 0$; see Debrunner and Flor [DF].

Browder [B3] obtained the following extension theorem of monotone sets:

Lemma. (Browder [B3, Theorem 8]) *Let K be a compact convex subset of a t.v.s. E, and F a t.v.s. with a bilinear pairing $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ which is continuous on compact subsets of $F \times E$. Let $f : K \to F$ be continuous and M a monotone subset of $K \times F$. Then there exists a $u_0 \in K$ such that*

$$\langle f(u_0) - w, u_0 - u \rangle \geq 0 \text{ for all } (u, w) \in M,$$

or equivalently, the set $M \cup \{(u_0, f(u_0))\}$ remains monotone.

This result sharpens corresponding result of Debrunner and Flor [DF] for E locally convex and of Fan [F, Theorem 12] for F locally convex and quasi-complete.

3. Better Admissible Multimaps and a Coincidence Theorem

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y having nonempty values. Note that $y \in T(x)$ is equivalent to $x \in T^-(y)$ and, for $B \subset Y$, let $T^-(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ and $T^+(B) := \{x \in X : T(x) \subset B\}$. For topological spaces X and Y, a map $T : X \multimap Y$ is *upper semicontinuous* if $T^+(B)$ is open for each open set B in Y; *lower semicontinuous* if $T^+(B)$ is closed for each closed set B in Y; and *continuous* if it is upper and lower semicontinuous.

A *convex space* X is a convex set (in a vector space) equipped with a topology that induces the Euclidean topology on convex hulls of each finite subset of X. Such convex hulls are called *polytopes*. See Lassonde [L].

Let X and Y be topological spaces. An *admissible class* $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is one such that, for each compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, Y)$ satisfying $\Gamma x \subset Tx$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact-valued; and
(iii) for each polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers map $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of topological vector spaces), admissible maps of Górniewicz, permissible maps of Dzedzej, and others. Further, $\mathbb{K}_c^\sigma$ due to Lassonde, $\mathbb{V}_c^\sigma$ due to Park *et al.*, and approximable maps $\mathbb{A}^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of $\mathfrak{A}_c^\kappa$. For the literature, see [P1], [P2].

We now define a new "better" admissible class defined on a convex space X :

$$F \in \mathfrak{B}(X,Y) \Longleftrightarrow \text{ for any polytope } P \text{ in } X \text{ and any } f \in \mathbb{C}(F(P),P),$$
$$f(F|_P) : P \multimap P \text{ has a fixed point.}$$

Note that $\mathfrak{A}_c^\kappa \subset \mathfrak{B}$ and some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were given in [P3].

The following coincidence theorem is a particular form of [P2, Theorem 1] and its proof is given here for the completeness:

Theorem 1. *Let X be a convex space, Y a Hausdorff space, and $T, S : X \multimap Y$ maps satisfying*

(1) *$T \in \mathfrak{B}(X,Y)$ is compact;*
(2) *for each $y \in T(X)$, $S^-(y)$ is convex; and*
(3) *$\{\operatorname{Int} S(x) : x \in X\}$ covers $\overline{T(X)}$.*

Then T and S have a coincidence point $x_0 \in X$; that is, $T(x_0) \cap S(x_0) \neq \emptyset$.

Proof. Since $\overline{T(X)}$ is compact and included in $\bigcup\{\operatorname{Int} S(x) : x \in X\}$, there exists an $N = \{x_1, x_2, \ldots, x_n\} \subset X$ such that $\overline{T(X)} \subset \bigcup\{\operatorname{Int} S(x) : x \in N\}$. Let $\{\lambda_i\}_{i=1}^n$ be the partition of unity subordinated to this cover of the Hausdorff compact space $\overline{T(X)}$, and $P = \operatorname{co} N \subset X$. Define $f : \overline{T(X)} \to P$ by

$$f(y) = \sum_{i=1}^n \lambda_i(y) x_i = \sum_{i \in N_y} \lambda_i(y) x_i$$

for $y \in \overline{T(X)} \subset Y$, where

$$i \in N_y \Longleftrightarrow \lambda_i(y) \neq 0 \Longrightarrow y \in \operatorname{Int} S(x_i) \subset S(x_i).$$

Then $x_i \in S^-(y)$ for each $i \in N_y$. Clearly f is continuous and, by (2), we have $f(y) \in \operatorname{co}\{x_i : i \in N_y\} \subset S^-(y)$ for each $y \in T(X)$. Since P is a polytope in X and $T \in \mathfrak{B}(X,Y)$, $(f|_{T(P)})(T|_P) : P \multimap P$ has a fixed point $x_0 \in P \subset X$. Since $x_0 \in (fT)(x_0)$ and $f^-(x_0) \subset S(x_0)$, we have $T(x_0) \cap S(x_0) \neq \emptyset$. This completes our proof.

Remark. For the subclass $\mathfrak{A}_c^\kappa$ of $\mathfrak{B}$, Theorem 1 is given earlier in [P1].

4. Extensions of Monotone Sets

We deduce the following equilibrium existence theorem from Theorem 1:

Theorem 2. *Let K be a compact convex subset of a t.v.s. E, K_1 a Hausdorff compact subset of a t.v.s. F, $T \in \mathfrak{B}(K, K_1)$ with closed graph, and $M \subset E \times F$. Let $\Phi : E \times F \to \mathbf{R} \cup \{-\infty\}$ be a function such that*

(1) *Φ is u.s.c. on compact subsets of $E \times F$;*
(2) *for each $x \in E$, $\Phi(x, \cdot)$ is l.s.c. on compact subsets of F;*
(3) *for each $w \in F$, $F(\cdot, w)$ is quasiconcave.*

Suppose that for each $y \in K_1$, there exists an $x \in K$ such that

$$\Phi(x - u, y - w) \geq 0 \quad \textit{for all } (u, w) \in M.$$

Then there exist a $u_0 \in K$ and a $w_0 \in T(u_0)$ such that

$$\Phi(u_0 - u, w_0 - w) \geq 0 \quad \textit{for all } (u, w) \in M.$$

Proof. For any $\varepsilon > 0$ and any nonempty finite subset N of M, we set

$$H_{(\varepsilon,N)} = \{(u_0, w_0) \in \mathrm{Gr}(T) : \Phi(u_0 - u, w_0 - w) \geq -\varepsilon \text{ for all } (u, w) \in N\}$$

and

$$\begin{aligned} H_0 &= \{(u_0, w_0) \in \mathrm{Gr}(T) : \Phi(u_0 - u, w_0 - w) \geq 0 \text{ for all } (u, w) \in M\} \\ &= \bigcap \{H_{(\varepsilon,N)} : \varepsilon > 0 \text{ and } N \text{ is a finite subset of } M\}. \end{aligned}$$

Then we have to show $H_0 \neq \emptyset$.

By (1), each $H_{(\varepsilon,N)}$ is a closed subset of $\mathrm{Gr}(T)$. The intersection of each finite family of such sets is also a set of the form $H_{(\varepsilon',N')}$ for some $\varepsilon' > 0$ and a finite subset N' of M. Therefore, in order to show $H_0 \neq \emptyset$, it suffices to show that each $H_{(\varepsilon,N)}$ is nonempty.

Choose a given $\varepsilon > 0$ and a nonempty finite subset N of M. Define a map $S : K \multimap K_1$ by

$$S(x) = \{y \in K_1 : \Phi(x-u, y-w) > -\varepsilon, \ (u,w) \in N\}$$

for $x \in X$. Then $S(x)$ is open in K_1 by (2). Moreover,

$$S^-(y) = \{x \in K : \Phi(x-u, y-w) > -\varepsilon, \ (u,w) \in N\}$$

is nonempty by hypothesis and convex by (3).

Now we apply Theorem 1. Then there exists a $(u_0, w_0) \in \mathrm{Gr}(T)$ such that $w_0 \in S(u_0)$; that is,

$$\Phi(u_0 - u, w_0 - w) > -\varepsilon \quad \text{for all } (u,w) \in N.$$

Therefore, $H_{(\varepsilon,N)}$ is nonempty. This completes our proof.

Remarks. 1. In Theorem 2, instead of $T \in \mathfrak{B}(K, K_1)$ with closed graph, we can adopt $T \in \mathfrak{A}_c^\kappa(K, K_1)$ without affecting its conclusion.

2. In Theorem 2, since T has closed graph and K_1 is compact, T itself is actually u.s.c. with compact values.

3. For the subclass $\mathbb{C}$ of $\mathfrak{B}$, Theorem 2 reduces to Fan [F, Theorem 11], who assumed that F is locally convex and other restrictions.

4. For the subclass $\mathbb{K}$ of $\mathfrak{B}$, Theorem 2 reduces to Browder [B3, Theorem 9], where F is locally convex.

The following is our theorem on extensions of monotone sets:

Theorem 3. *Let E be a t.v.s., F a Hausdorff t.v.s. with a bilinear pairing $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ which is continuous on compact subsets of $F \times E$, K a compact convex subset of E, and K_1 a compact subset of F. Let $T \in \mathfrak{B}(K, K_1)$ have closed graph and M a monotone subset of $K \times F$. Then there exist a $u_0 \in K$ and a $w_0 \in T(u_0)$ such that*

$$\langle w_0 - w, u_0 - u \rangle \geq 0 \quad \textit{for all } (u, w) \in M.$$

Proof. We put $\Phi(x, w) = \langle w, x \rangle$ for $(x, w) \in E \times F$. Then Φ satisfies conditions (1)–(3) in Theorem 2. By Theorem 2, it suffices to show that for each $y \in K_1$, there exists an $x \in K$ such that

$$\langle y - w, x - u \rangle \geq 0 \quad \text{for all } (u, w) \in M.$$

Now, we define $f : K \to K_1$ by

$$f(v) = y \quad \text{for all } v \in K.$$

By applying Lemma to f, such an $x \in K$ exists. This completes our proof.

Remarks. 1. In Theorem 3, we can replace $T \in \mathfrak{B}(K, K_1)$ with closed graph by $T \in \mathfrak{A}_c^\kappa(K, K_1)$.

2. For the subclass $\mathbb{C}$ of $\mathfrak{B}$, Theorem 2 reduces to Browder [B3, Theorem 8] or Lemma.

3. Even for the subclass $\mathbb{K}$ of $\mathfrak{B}$, Theorem 2 improves Browder [B3, Theorem 9], where F is assumed to be locally convex.

5. Noncompact Versions

The following is a noncompact version of Lemma :

Theorem 4. *Let X be a convex subset of a t.v.s. E, and F a t.v.s. with a bilinear pairing $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ which is continuous on compact subsets of $F \times E$. Let $f : X \to F$ be a function which is continuous on compact subsets of X, and $M \subset X \times F$ a monotone subset satisfying the following compactness condition:*

(0) *There is a nonempty compact subset K of X such that for each nonempty finite subset N of X, there exists a compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$ there exists a $(u, w) \in M \cap (L_N \times F)$ with $\langle f(x) - w, x - u \rangle < 0$.*

Then there exists a $u_0 \in X$ such that

$$\langle f(u_0) - w, u_0 - u \rangle \geq 0 \quad \textit{for all } (u, w) \in M.$$

Proof. For each $(u, w) \in M$, consider the set

$$K_{(u,w)} = \{x \in K : \langle f(x) - w, x - u \rangle \geq 0\}$$

which is a closed subset of K. We show that $\{K_{(u,w)} : (u, w) \in M\}$ has the finite intersection property. Let $\{(u_1, w_1), \ldots, (u_n, w_n)\}$ be a finite subset of M. Then there exists a compact convex subset L_N of X containing $N = \{u_1, \ldots, u_n\}$ as in condition (0). Applying Lemma with L_N instead of K, we obtain a $u_0 \in L_N$ such that

$$\langle f(u_0) - w, u_0 - u \rangle \geq 0 \quad \text{for all } (u, w) \in M \cap (L_N \times F).$$

Now by condition (0), we should have $u_0 \in K$. Therefore,

$$\bigcap_{i=1}^{n} K_{(u_i,w_i)} \supset \bigcap \{K_{(u,w)} : (u, w) \in M \cap (L_N \times F)\} \supset \{u_0\} \neq \emptyset.$$

Since K is compact, we have

$$\bigcap\{K_{(u,w)} : (u,w) \in M\} \neq \emptyset$$

This implies the conclusion.

Remraks. 1. For $X = K$, Theorem 4 reduces to Lemma or Browder [B3, Theorem 8].

2. Theorem 4 is a sharpened version of Lassonde [L, Theorem 2.7].

3. By adopting similar method, it is possible to establish noncompact versions of Theorems 3 and 4.

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