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Almost fixed points of multimaps having totally bounded ranges

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1. Introduction

The well-known Schauder conjecture is as follows: every continuous function, from a compact convex subset in a topological vector space into itself, would have a fixed point. This old conjecture is not yet resolved. For the history of partial solutions, see [2,4,7,17,24,25]. One of the most general partial solutions was given by Idzik [7] using the concept of convexly totally bounded (c.t.b.) sets. Actually, he gave a fixed point theorem for Kakutani maps $\Phi : X \multimap X$, where X is a convex subset of a Hausdorff topological vector space E, under the assumption that $\overline{\Phi(X)}$ is a compact c.t.b. subset of X. Since his ingenious proof is quite involved, it is desirable to give a transparent proof even for a particular case.

On the other hand, in proofs of many fixed point theorems, almost fixed points have usually appeared in an auxiliary role. In certain cases, almost fixed points, unlike fixed points, can be obtained numerically; and in some other cases, the existence of a fixed point is non-trivial or uncertain, whereas almost fixed points are easily found. Therefore, almost fixed points seem to be natural objects in many applications. For details, the reader can refer to Smart [23] and references therein.

In this paper, we give almost fixed point theorems for Kakutani maps or for a larger class of multimaps (so called, the better admissible class) having totally bounded

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ranges. Precisely, we assume that the closures of the ranges satisfy more restrictive conditions than that of c.t.b. sets. Our results are applied to obtain the most well-known fixed point theorems in analytical fixed point theory. Actually, our results include the historically well-known theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Hukuhara, Bohnenblust and Karlin, Fan, Glicksberg, Fort, Himmelberg, Granas and Liu, Lassonde, Smart, Chang and Yen, and others.

2. Preliminaries

We recall some related results of Idzik [7], De Pascale et al. [2], and Weber [24,25]. Throughout this paper, E denotes a topological vector space (t.v.s.).

A subset K of a t.v.s. E is said to be *of the Zima type*, by Hadžić [4], if for each neighborhood U of $0 \in E$ there exists a neighborhood V of $0 \in E$ such that

$$\mathrm{co}(V \cap (K - K)) \subset U.$$

A set $B \subset E$ is said to be c.t.b., by Idzik [7], if for every neighborhood V of $0 \in E$ there exist a finite subset $\{x_i\colon i \in I\} \subset B$ and a finite family of convex sets $\{C_i\colon i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \bigcup\{x_i + C_i\colon i \in I\}$.

Note that $\{x_i\colon i \in I\}$ can be chosen in E; see Idzik and Park [8].

Idzik [7] gave examples of c.t.b. sets:

(1) Every compact set in a locally convex t.v.s.

(2) Any compact set in a t.v.s. which is locally convex or is of the Zima type. Further, De Pascale et al. [2] gave other examples of c.t.b. sets:

(3) Every compact convex subset of $E = l_p$, $0 < p < 1$.

(4) More generally, every compact convex subset of a t.v.s. E on which its topological dual E' separates points.

For more examples, see [2,24,25].

Let $\overline{}$ denote the closure operation in a t.v.s. E.

The following is well known:

Theorem I. (Idzik [7, Theorem 4.3]). *Let X be a convex subset of a Hausdorff t.v.s. E and $\Phi : X \multimap X$ be a Kakutani map* (*that is, u.s.c. with nonempty compact convex values*). *If $\overline{\Phi(X)}$ is a compact c.t.b. subset of X, then there exists an $x \in X$ such that $x \in \Phi(x)$.*

Further, Idzik [7] raised the following question:

Problem I. Is every compact convex subset of a t.v.s. convexly totally bounded?

A positive answer to this question would resolve the Schauder conjecture. However, Idzik's problem was resolved negatively by the following:

Theorem D. (De Pascale et al. [12 Corollary 1.9]). *For $0 \leqslant p < 1$, the space $L_p(\mu)$, where μ denotes the Lebesgue measure on* $[0, 1]$, *contains compact convex subsets which are not c.t.b.*

Moreover, Weber [24,25] defined the following:

A subset K of a t.v.s. E is said to be *strongly convexly totally bounded* (s.c.t.b.) if for every neighborhood V of $0 \in E$ there exist a convex subset C of V and a finite subset N of K such that $K \subset N + C$.

The following is known:

Theorem W. (Weber [25, Corollary 2.8]). *Let K be a compact convex subset of a t.v.s. (E,τ) and $F = \operatorname{span} K$. Then the following conditions are equivalent*:

(1) *K is s.c.t.b.*
(2) *K is of Zima type.*
(3) *K is locally convex.*
(4) *K is affinely embeddable in a locally convex t.v.s.*
(5) *E admits a Hausdorff locally convex linear topology $\sigma = \sigma(E,E')$, which induces on F a finer topology than τ such that $\sigma|_K = \tau|_K$.*

Further, Weber [24] raised the following question:

Problem W. Is every convex c.t.b. set s.c.t.b.?

Now, the following form of the KKM principle is well known; see [9,13,17,19,20].

The KKM Principle. *Let X be a subset of a topological vector space D, a nonempty subset of X such that $\operatorname{co} D \subset X$, and $F : D \multimap X$ a multimap with closed [resp. open] values in X. If*

$$\operatorname{co} A \subset F(A) \tag{1}$$

for every nonempty finite subset A of D, then the family $\{F(x)\}_{x \in D}$ has the finite intersection property.

3. Almost fixed points

In this section, we obtain two almost fixed point theorems.

For a subset X of a t.v.s. E, a multimap $\Phi : X \multimap E$ is said to have *the (convexly) almost fixed point property* if for any (convex) neighborhood V of $0 \in E$, there exists an $x_V \in X$ such that $\Phi(x_V) \cap (x_V + V) \neq \emptyset$.

Our first almost fixed point theorem is the following:

Theorem 1. *Let X be a convex subset of a t.v.s. E and $\Phi : X \multimap E$ a u.s.c. multimap with convex values. If there is an s.c.t.b. subset K of X such that $\Phi(x) \cap K \neq \emptyset$ for each $x \in X$, then Φ has the almost fixed point property.*

Proof. For any neighborhood V of $0 \in E$, choose a symmetric open neighborhood U of 0 such that $\bar{U} \subset V$. Since K is s.c.t.b. in E, there exist a finite subset $\{x_1, x_2, \ldots, x_n\} \subset$

$K \subset X$ and a convex subset $C \subset U$ such that $K \subset \bigcup_{i=1}^n (x_i + C)$. For each i, let

$$F(x_i) := \{x \in X:\ \Phi(x) \cap (x_i + \bar{C}) = \emptyset\}.$$

Then each $F(x_i)$ is open since Φ is u.s.c. Moreover, we have

$$\bigcap_{i=1}^n F(x_i) = \left\{ x \in X:\ \Phi(x) \cap \bigcup_{i=1}^n (x_i + \bar{C}) = \emptyset \right\} = \emptyset$$

since $\emptyset \neq \Phi(x) \cap K \subset \Phi(x) \cap \bigcup_{i=1}^n (x_i + \bar{C})$ for each $x \in X$.

Now we apply the KKM principle with $D = \{x_i\}_{i=1}^n$. Since the conclusion of the principle for the open-valued case does not hold, condition (1) is violated. Hence, there exists a subset $N := \{x_{i_1}, \ldots, x_{i_k}\} \subset D$ and an $x_V \in \operatorname{co} N \subset X$ such that $x_V \notin F(N)$ or $\Phi(x_V) \cap (x_{i_j} + \bar{C}) \neq \emptyset$ for all $j = 1, \ldots, k$. Let L be the subspace of E generated by D, and

$$M := \{y \in L:\ \Phi(x_V) \cap (y + \bar{C}) \neq \emptyset\}.$$

Note that $N \subset M$. Since L, $\Phi(x_V)$, and $\bar{C}$ are all convex, it is easily checked that M is convex. Therefore, $x_V \in \operatorname{co} N \subset \operatorname{co} M = M$ and, by definition of M, we get $\Phi(x_V) \cap (x_V + \bar{C}) \neq \emptyset$. This shows that $\Phi(x_V) \cap (x_V + V) \neq \emptyset$. This completes our proof. □

Corollary 1.1. *Let K be a compact convex subset of a t.v.s. satisfying one of the conditions* (1)–(5) *of Theorem* W. *Let $\Phi : K \multimap E$ be a u.s.c. multimap with convex values such that $\Phi(x) \cap K \neq \emptyset$ for each $x \in X$. Then Φ has the almost fixed point property.*

Let X be a convex subset of a vector space and Y a topological space. Motivated by our earlier work, Chang and Yen [1] defined the following:

$T \in \mathrm{KKM}(X, Y) \Leftrightarrow T : X \multimap Y$ is a multimap such that the family $\{S(x): x \in X\}$ has the finite intersection property whenever $S : X \multimap Y$ has closed values and $T(\operatorname{co} A) \subset S(A)$ for each nonempty finite subset A of X.

The following is our second main result of this section.

Theorem 2. *Let X be a convex subset of a t.v.s. E and $\Phi \in \mathrm{KKM}(X, \bar{X})$. If $\overline{\Phi(X)}$ is totally bounded, then Φ has the convexly almost fixed point property.*

Proof. For any convex neighborhood V of $0 \in E$, we have an open convex neighborhood C of 0 such that $C \subset V$ and a nonempty finite subset $\{x_1, x_2, \ldots, x_n\} \subset \overline{\Phi(X)}$ such that $\overline{\Phi(X)} \subset \bigcup_{i=1}^n (x_i + C)$. We may assume that $\{x_1, x_2, \ldots, x_n\} \subset X$. In fact, let U be a symmetric neighborhood of $0 \in E$ such that $U + U \subset C$. Suppose that $\{y_1, y_2, \ldots, y_n\} \subset \overline{\Phi(X)} \subset \bar{X}$ and D is an open convex neighborhood of 0 such that $\overline{\Phi(X)} \subset \bigcup_{i=1}^n (y_i + D)$ and $D \subset U$. Since $\{y_i + D\}_{i=1}^n$ is an open cover of $\overline{\Phi(X)} \subset \bar{X}$, we have $(y_i + D) \cap X \neq \emptyset$ for each i. Choose an $x_i \in X \cap (y_i + D)$ for each i. Then $\overline{\Phi(X)} \subset \bigcup_{i=1}^n (x_i + (y_i - x_i) + D)$ and the open convex set $y_i - x_i + D \subset -D + D \subset$

$U + U \subset C$. Then, $\overline{\Phi(X)} \subset \bigcup_{i=1}^n (x_i + C)$ and $x_i \in X$ for each i. Let us define a multimap $F : X \multimap \bar{X}$ by

$$F(x) := \overline{\Phi(X)} \setminus (x + C) \quad \text{for each } x \in X.$$

Then F is closed-valued and

$$\bigcap_{i=1}^n F(x_i) = \overline{\Phi(X)} \setminus \bigcup_{i=1}^n (x_i + C) = \emptyset.$$

Since $\Phi \in \mathrm{KKM}(X, \bar{X})$ and $\{F(x) : x \in X\}$ does not have the finite intersection property, we have $\Phi(\mathrm{co}\, A) \not\subset F(A)$ for a nonempty finite subset $A \subset X$. Therefore, there exist $x_V \in \mathrm{co}\, A \subset X$ and $y_V \in \Phi(x_V) \subset \overline{\Phi(X)}$ such that

$$y_V \notin F(z) = \overline{\Phi(X)} \setminus (z + C) \quad \text{for all } z \in A.$$

Therefore, $y_V \in z + C$ for all $z \in A$. Let $A := \{z_1, z_2, \ldots, z_m\}$ and $x_V := \sum_{j=1}^m \lambda_j z_j$, where $\lambda_j \geqslant 0$ and $\sum_{j=1}^m \lambda_j = 1$. Then $y_V = (\sum_{j=1}^m \lambda_j) y_V \in \sum_{j=1}^m \lambda_j z_j + \sum_{j=1}^m \lambda_j C \subset x_V + C \subset x_V + V$. Therefore, $y_V \in \Phi(x_V) \cap (x_V + V) \neq \emptyset$. This completes our proof. □

Note that a particular form of Theorem 2 was obtained by Chang and Yen [1].

From Theorem 2, we have the following corollary:

Corollary 2.1. *Let X be a convex subset of a locally convex t.v.s. E and $f : X \to \bar{X}$ a continuous map such that $\overline{f(X)}$ is totally bounded. Then f has the almost fixed point property.*

Proof. Note that a continuous map $f : X \to \bar{X}$ belongs to $\mathrm{KKM}(X, \bar{X})$. □

We remark that we can derive Corollary 2.1 directly from the KKM principle just by following the proof of Theorem 1.

Corollary 2.2. *Let X be a convex subset of a locally convex t.v.s. E and $f : X \to X$ a continuous map. If X is totally bounded, then f has the almost fixed point property.*

Proof. Note that $f : X \to X$ can be regarded as $f : X \to \bar{X}$. Since $f(X) \subset X$ is totally bounded, so is $\overline{f(X)}$. Now, the conclusion follows from Corollary 2.1. □

Example. (1) If E is a metric t.v.s. whose balls are convex, then Corollaries 2.1 and 2.2 hold.

Note that Smart [23] obtained Corollaries 2.1 and 2.2 for the following cases:

(2) E is a normed vector space.

(3) X is an open ball in a normed vector space.

(4) X is an open ball in $\mathbb{R}^n$.

Earlier, Fort [3] obtained Corollary 2.2 when

(5) X is an open disk in $\mathbb{R}^2$.

Note that Theorem 2 and Corollaries 2.1 and 2.2 do not guarantee the existence of fixed points of Φ or f.

Example. Let $X=\{(x,y): x^2+y^2<1\}$ be the open unit disk in $\mathbb{R}^2$ and $f: X \to \bar{X}$ a continuous map such that $f(x,y)=(x,\sqrt{1-x^2})$ for all $(x,y)\in X$.

4. Fixed point theorems

In this section, we show that Theorems 1 and 2 can be applied to obtain fixed point theorems.

Recall that, for topological spaces X and Y, a multimap $T: X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T)=\{(x,y): x\in X, y\in T(x)\}$ is closed in $X\times Y$, and *compact* if the range $T(X)$ is contained in a compact subset of Y.

Lemma 1. *Let X be a subset of a Hausdorff t.v.s. E, K a compact subset of X, and $\Phi: X \multimap X$ a closed map with nonempty values. If, for any neighborhood V of $0\in E$, there exists an $x_V\in X$ such that $K\cap\Phi(x_V)\cap(x_V+V)\neq\emptyset$, then Φ has a fixed point $x_0\in K$; that is, $x_0\in\Phi(x_0)$.*

Proof. For each neighborhood V of $0\in E$, there exists a $y_V\in K$ such that $y_V\in\Phi(x_V)\cap(x_V+V)$. Since K is compact, we may assume that the net $\{y_V\}$ converges to some $x_0\in K\subset X$. Since X is Hausdorff and $y_V\in x_V+V$, the net $\{x_V\}$ also converges to x_0. Since $(x_V,y_V)\in\mathrm{Gr}(\Phi)$ and $\mathrm{Gr}(\Phi)$ is closed, we have $(x_0,x_0)\in\mathrm{Gr}(\Phi)$. This completes our proof. □

We have the following from Theorem 1 and Lemma 1:

Theorem 3. *Let X be a convex subset of a Hausdorff t.v.s. E and $\Phi: X\multimap X$ a compact u.s.c. multimap with nonempty closed convex values. If $\overline{\Phi(X)}$ is an s.c.t.b. subset of X, then Φ has a fixed point $x_0\in X$.*

Proof. Note that Φ is closed and $K:=\overline{\Phi(X)}$ is a compact subset of X. Any closed compact multimap having the almost fixed point property has a fixed point. □

Note that Theorem 3 is a particular case of Theorem I of Idzik. However, our proof is based on the KKM principle only, and is more easily accessible.

Corollary 3.1. *Let K be a compact convex subset of a Hausdorff t.v.s. satisfying one of the conditions* (1)–(5) *of Theorem* W. *Then any Kakutani map $\Phi: K\multimap K$ has a fixed point.*

Note that Corollary 3.1 includes the well-known results due to Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Granas and Liu, and Park; for the literature, see [11,13,16,17].

From Theorem 2, we deduce fixed point theorems for the better admissible class of closed compact multimaps $\Phi: X\multimap X$, where X is a convex subset of a locally convex t.v.s.

Let X be a nonempty convex subset of a t.v.s. E. A *polytope* P in X is any convex hull of a nonempty finite subset of X, or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

For topological spaces X and Y, an *admissible* class $\mathfrak{A}_c^\kappa(X,Y)$ of maps $F: X \multimap Y$ is one such that, for each F and each nonempty compact subset K of X, there exists a map $G \in \mathfrak{A}_c(X,Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $T \in \mathfrak{A}_c$ is upper semicontinuous (u.s.c.) with nonempty compact values; and
(iii) for any polytope P, each $T \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces are suitably chosen.

The "*better*" admissible class $\mathfrak{B}$ of multimaps defined from a convex set X to a topological space Y is defined as follows:

$F \in \mathfrak{B}(X,Y) \Leftrightarrow F: X \multimap Y$ is a multimap such that for any polytope P in X and any continuous map $f: F(P) \to P$, $f \circ (F|_P): P \multimap P$ has a fixed point.

Subclasses of $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps (u.s.c. with nonempty compact convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (u.s.c. with R_δ values), the acyclic maps $\mathbb{V}$ (u.s.c. with compact acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ-selectional maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^+$ of Lassonde, the class $\mathbb{V}_c^+$ of Park et al. [21], and approximable maps of Ben-El-Mechaiekh and Idzik, and many others. These subclasses are all examples of the admissible class $\mathfrak{A}_c^\kappa$. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ are known. For details on the better admissible classes, see [12–18].

The following is known [12,14]:

Lemma 2. *Let X be a convex subset of a t.v.s. E and Y a Hausdorff space. Then*
(1) $\mathfrak{A}_c^\kappa(X,Y) \subset \mathrm{KKM}(X,Y)$; *and*
(2) *in the class of closed compact multimaps, two subclasses* $\mathfrak{B}(X,Y)$ *and* $\mathrm{KKM}(X,Y)$ *coincide.*

It should be noted that there are only a few trivial examples of maps in $\mathrm{KKM}(X,Y)$ which are not in $\mathfrak{A}_c^\kappa$ or $\mathfrak{B}$; see [1].

From Theorem 2 and Lemma 2, we have the following:

Theorem 4. *Let X be a convex subset of a locally convex Hausdorff t.v.s. E and $\Phi \in \mathfrak{B}(X,X)$. If Φ is closed and compact, then Φ has a fixed point.*

Proof. Any closed compact multimap having the almost fixed point property has a fixed point. Therefore, Theorem 4 follows from Theorem 2 and Lemma 2. □

Comparing Theorem 4 with Theorem I, we have a fixed point theorem for a much more general class of multimaps under a more restrictive condition on the space itself than Idzik's.

Theorem 4 was obtained by the author [14], and contains fixed point theorems due to Himmelberg [5], Lassonde [10], Park [11], Park et al. [21], Chang and Yen [1], and many others; see [15,16]. Consequently, we have given different proofs of those known results.

In view of Theorem W, we have the following:

Corollary 4.1. *Let K be a compact convex subset of a Hausdorff t.v.s. and $\Phi\in\mathfrak{B}(K,K)$ a closed multimap. Then Φ has a fixed point if one of the following equivalent conditions hold*:

(1) *K is s.c.t.b.*

(2) *K is of Zima type.*

(3) *K is locally convex.*

A simple example of Theorem 4 is as follows:

Corollary 4.2. *Let X be a convex subset of a locally convex t.v.s. E and $f : X \to X$ a continuous compact map. Then f has a fixed point.*

Corollary 4.2 was obtained by Hukuhara [6] with a different proof, and includes fixed point theorems due to Brouwer (for an n-simplex X), Schauder (for a normed vector space E), and Tychonoff (for a compact convex subset X).

We have one more

Corollary 4.3. *Let X be a convex subset of a metric t.v.s. E whose balls are convex and $f : X \to X$ a continuous compact map. Then f has a fixed point.*

If X itself is compact, then Corollary 4.3 reduces to a result of Rassias [22].

Note that a compact convex subset K satisfying one of (1)–(3) in Corollary 4.1 is admissible in the sense of Klee; see [25, Theorem 1.9].

Finally, we note that a far-reaching generalization of Theorem 4 was already obtained by the author [14,15] as follows:

Theorem P. *Let X be an admissible* (*in the sense of Klee*) *convex subset of a Hausdorff t.v.s. E. Then any closed compact map $\Phi\in\mathfrak{B}(X,X)$ has a fixed point.*

In [16], we have listed more than sixty papers in chronological order, from which we could deduce particular forms of Theorem P.

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