GENERALIZED KIRSZBRAUN–MINTY TYPE INEQUALITIES

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Abstract. A generalized Kirszbraun–Minty type inequality theorem is deduced from a Ky Fan type minimax inequality due to the author. Our new theorem subsumes known results of Kirszbraun, Valentine, Minty, Grünbaum, Debrunner and Flor, Kassay and Kolumban.

1. Introduction

The well-known Kirszbraun theorem [K] asserts that a nonexpansive function from a finite domain in $\mathbf{R}^n$ to $\mathbf{R}^n$ can be extended to a larger domain including any arbitrarily chosen point so as to be nonexpansive. Extensions and variations of the theorem were obtained by Valentine [V1,2], Minty [M1-4], Grünbaum [G2], Debrunner and Flor [DF], and others with different proofs. Moreover, applications to certain problems were followed; see Minty [M4] and the references in the present paper.

Recently, Kassay and Kolumban [KK] obtained some necessary and sufficient conditions for generalizations of Minty's inequality [M4]. Consequently, their results include extensions of theorems of Kirszbraun [K], Grünbaum [G2], and Minty [M4]. Moreover, they also obtained another generalization of Minty's inequality [M4] by applying the well-known Ky Fan minimax inequality.

2000 *Mathematics Subject Classification.* 49J99.

Key words and phrases. Convex space, quasiconcave, lower semicontinuous (l.s.c.), γ-diagonally quasiconcave, finitely l.s.c., vector space, t.v.s.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

In the present paper, we obtain a generalized Kirszbraun–Minty type inequality theorem which subsumes all of the above-mentioned results of the same sort. Our argument is based on a modern version of the Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM theorem) due to Lassonde [L], from which we deduce our own version of a minimax inequality or an equilibrium theorem (Theorem 1). Our main result (Theorem 2) is given in Section 2; and Section 3 devotes to show particular forms of our main result and to indicate some applications. Finally, in Section 4, further remarks on our main result are added.

2. Main result

A *convex space* X is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets; see Lassonde [L].

We begin with the following form of the KKM theorem due to Lassonde [L]:

Theorem 0. *Let D be any subset of a convex space X and $G : D \multimap X$ a KKM multimap [that is, $\mathrm{co}\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$ for each finite $\{x_1, \cdots, x_n\} \subset D$] with closed values. If $G(x)$ is compact for at least one $x \in D$, then*

$$\bigcap\{G(x) : x \in D\} \neq \emptyset.$$

A real-valued function $f : X \to \mathbf{R}$ is said to be *quasiconcave* if $\{x \in X : fx > \gamma\}$ is convex for all $\gamma \in \mathbf{R}$; and *lower semicontinuous* (*l.s.c.*) if $\{x \in X : fx \leq \gamma\}$ is closed for all $\gamma \in \mathbf{R}$.

Let Δ_m be the standard $(m-1)$-simplex; that is,

$$\Delta_m = \{\lambda \in \mathbf{R}^m : \lambda = (\lambda_1, \ldots, \lambda_m),\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1\}.$$

From Theorem 0, we deduce the following equilibrium result:

Theorem 1 *Let X be a compact convex space, $\phi : X \times X \to \mathbf{R}$ an extended real-valued function, and $\gamma \in \mathbf{R}$ such that*

(i) *for each $x \in X$, $\{y \in X : \phi(x,y) \leq \gamma\}$ is closed*;

(ii) *for each $y \in X$, $\{x \in X : \phi(x,y) > \gamma\}$ is convex*; *and*

(iii) *for each $x \in X$, $\phi(x,x) \leq \gamma$.*

Then there exists a $\hat{y} \in X$ such that

$$\phi(x,\hat{y}) \leq \gamma \quad \textit{for all} \quad x \in X.$$

Proof. We apply Theorem 0 with $X = D$. Define a multimap $G : X \multimap X$ by $G(x) := \{y \in X : \phi(x,y) \leq \gamma\}$ for $x \in X$. Then each $G(x)$ is closed by (i). Moreover, each $G(x)$ is compact since X is compact. We claim that G is a KKM map. Suppose, on the contrary, that there exists a finite subset $\{x_1, \cdots, x_n\}$ of X such that $\text{co}\{x_1, \cdots, x_n\} \not\subset \bigcup_{i=1}^n G(x_i)$. Then there exists a $y \in \text{co}\{x_1, \cdots, x_n\}$ such that $y \notin G(x_i)$ for each i, or equivalently, $\phi(x_i, y) > \gamma$ for each i. Since $\{x \in X : \phi(x,y) > \gamma\}$ is convex by (ii) and contains $\{x_i\}_{i=1}^n$, we have

$$y \in \text{co}\{x_1, \cdots, x_n\} \subset \{x \in X : \phi(x,y) > \gamma\}$$

and hence $\phi(y,y) > \gamma$. This contradicts (iii). Therefore, by Theorem 0, we have $\bigcap\{G(x) : x \in X\} \neq \emptyset$. Hence, there exists a $\hat{y} \in X$ such that $\hat{y} \in G(x)$ for all $x \in X$, or equivalently, $\phi(x,\hat{y}) \leq \gamma$ for all $x \in X$. This completes our proof.

Remarks. 1. Theorem 1 is a particular form of results due to the author in [P1], [P2], [PK], and equivalent to the well-known Ky Fan minimax inequality or the Fan–Browder fixed point theorem; for the literature, see [P3].

2. If $\phi(\cdot, y)$ is quasiconcave, then (ii) holds; and if $\phi(x, \cdot)$ is l.s.c., then (i) holds.

From Theorem 1, we deduce the following main result of this paper:

Theorem 2. *Let X be a vector space, Y a nonempty set, $\Phi : X \times Y \times Y \to \mathbf{R}$ a function such that*

(0) *for each $y, y' \in Y$, $\Phi(\cdot, y, y')$ is finitely l.s.c.* (*that is, it is l.s.c. on any finite dimensional subspace of X*).

Let $(x_1, y_1), \cdots, (x_m, y_m) \in X \times Y$ *and* $y \in Y$ *be given. Suppose that*

(I) $\sum_{i=1}^{m} \lambda_i \Phi(x_i - x, y_i, y) \leq 0$ *for all* $x = \sum_{j=1}^{m} \lambda_j x_j$ *with* $\lambda \in \Delta_m$.

Then we have the following:

(II) *There exists an* $x \in \mathrm{co}\{x_1, \ldots, x_m\}$ *such that*

$$\Phi(x_i - x, y_i, y) \leq 0 \quad \textit{for each} \quad 1 \leq i \leq m.$$

Proof. Let $h : \Delta_m \times \Delta_m \to \mathbf{R}$ be given by

$$h(\lambda, \mu) = \sum_{i=1}^{m} \lambda_i \Phi(x_i - \sum_{j=1}^{m} \mu_j x_j,\ y_i, y)$$

for $(\lambda, \mu) \in \Delta_m \times \Delta_m$. Then for each $\lambda \in \Delta_m$, $\mu \mapsto h(\lambda, \mu)$ is l.s.c. on Δ_m by (0); and for each $\mu \in \Delta_m$, the set $\{\lambda \in \Delta_m : h(\lambda, \mu) > 0\}$ is convex. Indeed, it is clear that if $h(\lambda^1, \mu) > 0$ and $h(\lambda^2, \mu) > 0$, then $h(t\lambda^1 + (1-t)\lambda^2, \mu) > 0$ for $t \in [0, 1]$. Moreover, (I) implies $h(\lambda, \lambda) \leq 0$ for all $\lambda \in \Delta_m$. Therefore, by Theorem 1, there exists a $\mu_0 \in \Delta_m$ such that

$$h(\lambda, \mu_0) \leq 0 \quad \text{for all} \quad \lambda \in \Delta_m.$$

By choosing λ any of

$$(1, 0, \ldots, 0),\ (0, 1, \ldots, 0),\ \ldots\ ,\ (0, \ldots, 0, 1),$$

we have (II).

This completes our proof.

3. Particular forms and applications

In this section, we show that Theorem 2 extends a number of known Kirszbraun–Minty type theorems, and indicate applications of some of them. Our list is given in chronological order.

Kirszbraun [K]: $X = Y = \mathbf{R}^n$ and $\Phi(x_i - x, y_i, y) = \|x_i - x\| - \|y_i - y\|$. The Euclidean norm is essential; see Grünbaum [G1] and Schönbeck [S].

Valentine [V1,2]: Rediscovered the above result by applying the KKM theorem, and applied to extension problems of a vector function so as to preserve a Lipschitz condition.

Mickle [M], Schoenberg [S]: New proofs of the Kirszbraun theorem were given. Mickle used certain quadratic forms characterizing Euclidean spaces.

Minty [M1]: X is a t.v.s., Y the vector space of real continuous linear functionals over X, and

$$\Phi(x_i - x, f_i, f) = (f_i - f)(x_i - x).$$

Then for each $x \in X$ [resp., for each $f \in Y$] there exists an $f \in Y$ [resp., an $x \in X$] such that $(f_i - f)(x_i - x) \geq 0$ for all i. Minty actually gave a particular form.

Minty [M2]: For a Hilbert space X and an index set A, Kirszbraun's theorem is extended to any $\{(x_i, y_i)\}_{i \in A}$. Applied this result to problems on monotone operators.

Grünbaum [G2]: $X = Y = \mathbf{R}^n$ and

$$\Phi(x_i - x, y_i, y) = k_1(\|x_i - x\|^2 - \|y_i - y\|^2) + k_2\langle x_i - x, y_i - y\rangle$$

with nonnegative $k_1,\ k_2$. The results in [M1] and [G2] have interesting applications to nonlinear elliptic boundary value problems (see [B1]) and monotone operator theory (for example, see [B2], [Z]).

Debrunner and Flor [DF]: X and Y are t.v.s. and $\Phi(x_i - x, y_i, y) = \langle x_i - x, y_i - y\rangle$ is bilinear. The proof is based on the Fan–Glicksberg fixed point theorem; see

[P3]. The result generalizes earlier works of Minty [M1] and Grünbaum [G2]. An application to extensions of monotone sets is given.

Minty [M3]: $X = V$ and $Y = V^1$ are vector spaces, $\langle\ ,\ \rangle : V \times V^1 \to \mathbf{R}$ a bilinear form, and

$$\Phi(x_i - x, y_i, y) = \langle x_i - x, y_i - y\rangle.$$

In fact, $\langle\ ,\ \rangle : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ is assumed to be continuous and Minty obtained the conclusion for $y = 0$. Applications to calculus of variations and to maximal monotone sets are added.

Minty [M4]: X is a vector space, Φ is a Kirszbraun function; that is, $\Phi(\cdot, y, y')$ is finitely l.s.c. and convex, and satisfies

$$\sum_{i,j=1}^{m} \lambda_i \lambda_j \Phi(x_i - x_j, y_i, y_j) \geq 2 \sum_{i=1}^{m} \lambda_i \Phi(x_i - x, y_i, y)$$

for any $(x_1, y_1), \cdots, (x_m, y_m) \in X \times Y$, $y \in Y$, and $\lambda \in \Delta_m$, where $x = \sum_{j=1}^{m} \lambda_j x_j$. Then Φ satisfies condition (0) of Theorem 2. He further assumed $\Phi(x_i - x_j, y_i, y_j) \leq 0$ for all i, j. Consequently, Φ satisfies condition (I) of Theorem 2. He deduced (II) by applying the von Neumann minimax theorem. He also gave a lot of examples of Kirszbraun functions, and added applications to extension theorems for Lipschitz or Lipschitz-Hölder continuous functions.

Kassay and Kolumban [KK, Theorem 2.5]: The following particular form of condition (I) is assumed to deduce (II) in Theorem 2:

(I)$'$ for each $\lambda \in \Delta_m$ and $x = \sum_{j=1}^{m} \lambda_j x_j$,

$$0 \geq \sum_{i,j=1}^{m} \lambda_i \lambda_j \Phi(x_i - x_j, y_i, y_j) \geq k \sum_{i=1}^{m} \lambda_i \Phi(x_i - x, y_j, y),$$

where k is a positive constant which may depend on $\{(x_i, y_i)\}_{1 \leq i \leq m}$.

4. Remarks

Each of conditions (I) and (II) readily implies the following:

(III) $\min_{\lambda \in \Delta_m} \sum_{i=1}^{m} \lambda_i \Phi(x_i - x, y_i, y) \le 0$ for $x = \sum_{j=1}^{m} \lambda_j x_j$;

or more generally,

(III)$'$ for each $\lambda \in \Delta_m$, $\min_{\mu \in \Delta_m} \sum_{i=1}^{m} \lambda_i \Phi(x_i - x, y_i, y) \le 0$ for $x = \sum_{j=1}^{m} \mu_j x_j$.

Therefore, it is natural to ask whether (III) or (III)$'$ implies (II) in Theorem 2.

Kassay and Kolumban [KK, Theorem 2.4] showed that (II) is equivalent to (III)$'$ for the case $\Phi : X \times Y \to \mathbf{R}$ in Theorem 2 under the following additional restriction:

(IV) For each finite set $A = \{a_1, \ldots, a_k\} \subset X$ and subset $\{(x_{i_1}, y_{i_1}), \ldots, (x_{i_p}, y_{i_p})\}$ of $\{(x_i, y_i)\}_{1 \le i \le m}$, we have

$$\max_{1 \le j \le p} \sum_{s=1}^{k} \lambda_s \Phi(x_{i_j} - a_s, y_{i_j}) \ge \inf_{a \in A} \max_{1 \le j \le p} \Phi(x_{i_j} - a, y_{i_j})$$

for each $\lambda \in \Delta_k$.

They further gave an example of Φ satisfying condition (IV).

However, in general, (III) does not imply (II). For example, let $X = \mathbf{R}$, $Y = (0, \infty)$, and $\Phi : X \times Y \times Y \to \mathbf{R}$ be given by $\Phi(x, y, y') = |x|^y$. Then Φ is continuous in x and hence condition (0) is satisfied. But if $m = 2$, $(x_1, y_1) = (0, 1)$, and $(x_2, y_2) = (1, 2)$, then condition (III) holds, but condition (II) fails. This example is supplied by a reviewer of the original version of this paper. The present author would like to express his gratitude to the reviewer.

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