



ACADEMIC  
PRESS

J. Math. Anal. Appl. 269 (2002) 244–255

*Journal of*  
MATHEMATICAL  
ANALYSIS AND  
APPLICATIONS

www.academicpress.com

# Remarks on equilibria for $g$ -monotone maps on generalized convex spaces

Sehie Park<sup>1</sup>

*Department of Mathematics, Seoul National University, Seoul 151-742, South Korea*

Received 28 November 2000; accepted 16 November 2001

Submitted by C.E. Chidume

---

## Abstract

Recently, Oettli and Schläger obtained existence results for a class of vectorial equilibrium problems involving  $g$ -monotone multimaps. In this paper, we obtain sharpened generalizations of their results. We are based on the KKM theorems for generalized convex spaces and fixed point theorems for the better admissible class of multimaps. © 2002 Elsevier Science (USA). All rights reserved.

---

## 1. Introduction

Let  $X$  be a convex subset in a topological vector space (t.v.s.)  $E$ , and  $f : X \times X \rightarrow \mathbb{R}$  a given function such that  $f(x, x) \geq 0$  for all  $x \in X$ . The scalar equilibrium problem in the sense of Blum and Oettli [1] deals with the existence of

$$\bar{x} \in X \quad \text{such that} \quad f(\bar{x}, y) \geq 0 \quad \forall y \in X. \quad (1)$$

This problem subsumes in particular optimization problems, Nash equilibria in noncooperative games, and variational inequalities. In certain cases, the function  $f(\cdot, \cdot)$  is assumed to be pseudomonotone [2] or maximal pseudomonotone [3]. In order to unify the general case and the pseudomonotone case, Oettli and

---

*E-mail address:* shpark@math.snu.ac.kr (S. Park).

<sup>1</sup> Supported in part by the Brain Korea 21 Project.

Schläger [4] introduced another function  $g: X \times X \rightarrow \mathbb{R}$  and defined the  $g$ -monotonicity and the maximal  $g$ -monotonicity of  $f$ .

Moreover, in 1998, Oettli and Schläger [5] obtained existence results for a class of vectorial equilibrium problems involving  $g$ -monotone multimaps. This was carried out as follows: there is given a multimap  $f: X \times X \multimap Z$ ,  $Z$  a real t.v.s. For each  $x \in X$  there is given a closed convex cone  $P(x) \subset Z$  with  $P(x) \neq Z$  and  $\text{int } P(x) \neq \emptyset$ . (We use  $\subset$  instead of  $\subseteq$  in [4,5].) Then one asks for the existence of

$$\bar{x} \in X \quad \text{such that} \quad f(\bar{x}, y) \not\subset -\text{int } P(\bar{x}) \quad \forall y \in X. \quad (2)$$

For technical reasons the following notion of  $g$ -monotonicity is used: for all  $x, y \in X$ ,

$$f(x, y) \not\subset -\text{int } P(x) \quad \Rightarrow \quad g(x, y) \subset -P(y), \quad (3)$$

where  $g: X \times X \multimap Z$  is another multimap. The associated maximal  $g$ -monotonicity requires, in addition, that for all  $x \in X$ ,

$$g(x, y) \subset -P(y) \quad \forall y \in X \quad \Rightarrow \quad f(x, y) \not\subset -\text{int } P(x) \quad \forall y \in X. \quad (4)$$

It is interesting to note that the convex cones  $-\text{int } P(x)$  and  $-P(y)$  occurring in (2)–(4) can be replaced by more general sets  $C(x)$  and  $D(y)$ . This was done in Theorem 1 of [5]. Then, in Corollary 1 of [5], the requirement (4) was replaced by other, more practical, conditions.

In the second part of [5], a combination of equilibria and variational inequalities, called mixed equilibria, was considered: Given an additional multimap  $T: X \multimap L(E, Z)$ , we are concerned with the existence of

$$\begin{aligned} \bar{x} \in X \quad \text{and} \quad \bar{\xi} \in T(\bar{x}) \quad \text{such that} \\ f(\bar{x}, y) + \langle \bar{\xi}, y - \bar{x} \rangle \not\subset -\text{int } P(x) \quad \forall y \in X. \end{aligned} \quad (5)$$

The monotonicity requirement in this case does not refer to the map  $f(x, y)$  itself, but rather to the family of mappings  $f(x, y) + \langle \xi, y - x \rangle$  for all  $\xi \in T(X)$ .

It should be noticed that the main tools in [5] were the KKM theorem due to Ky Fan and the Fan–Glicksberg fixed point theorem. Those theorems are extended by the author to the KKM theorems for generalized convex spaces [6–9] and to fixed point theorems for the better admissible class of multimaps [7,10–12]. Therefore, from our new theorems, we can obtain sharpened generalizations of the whole results in [5]. Our main aim in this paper is to obtain such generalizations containing much more particular cases and, consequently, we show that theoretical grounds are much deeper than what we have thought.

## 2. Generalized convex spaces and the KKM theorems

We recall the following in [6–14]:

A *generalized convex space* or a *G-convex space*  $(X, D; \Gamma)$  consists of a topological space  $X$  and a nonempty set  $D$  such that for each  $N = \{z_0, z_1, \dots, z_n\} \subset D$ , there exist a subset  $\Gamma(N) = \Gamma_N$  of  $X$  and a continuous function  $\phi_N : \Delta_n \rightarrow \Gamma(N)$  such that  $J \subset \{0, 1, \dots, n\}$  implies  $\phi_N(\Delta_J) \subset \Gamma(\{z_j : j \in J\})$ , where  $\Delta_n$  is an  $n$ -simplex with vertices  $v_0, v_1, \dots, v_n$  and  $\Delta_J = \text{co}\{v_j : j \in J\}$  the face of  $\Delta_n$  corresponding to  $J$ .

In case to emphasize  $X \supset D$ ,  $(X, D; \Gamma)$  will be denoted by  $(X \supset D; \Gamma)$ ; and if  $X = D$ , then  $(X; \Gamma) := (X, X; \Gamma)$ .

There are a large number of examples of  $G$ -convex spaces. Typical examples are convex subsets of a t.v.s., convex spaces in the sense of Lassonde, and  $C$ -spaces due to Horvath. For other examples, see [6–9,13] and references therein.

For a  $G$ -convex space  $(X, D; \Gamma)$ , a multimap  $F : D \multimap X$  is called a *KKM map* if  $\Gamma_N \subset F(N)$  for each  $N \in \langle D \rangle$ , where  $\langle D \rangle$  denotes the set of all nonempty finite subsets of  $D$ .

The following due to Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) and others is well-known:

**The KKM Principle.** Let  $D$  be the set of vertices of an  $n$ -simplex  $\Delta_n$  and  $F : D \multimap \Delta_n$  be a KKM map (that is,  $\text{co } N \subset F(N)$  for each  $N \in \langle D \rangle$ ) with closed (resp. open) values. Then  $\bigcap_{z \in D} F(z) \neq \emptyset$ .

For a multimap  $F : D \multimap X$ , we define a multimap  $\overline{F} : D \multimap X$  by  $\overline{F}(z) := \overline{F(z)}$  for all  $z \in D$ , where  $\overline{\phantom{x}}$  denotes the closure operator.

The following is a KKM theorem for  $G$ -convex spaces due to the author [6,8,9]:

**Theorem 2.1.** Let  $(X, D; \Gamma)$  be a  $G$ -convex space and  $F : D \multimap X$  a multimap such that

- (1)  $F$  has closed (resp. open) values; and
- (2)  $F$  is a KKM map.

Then  $\{F(z)\}_{z \in D}$  has the finite intersection property.

Furthermore, if

- (3)  $\bigcap_{z \in M} \overline{F}(z)$  is compact for some  $M \in \langle D \rangle$ ,

then we have

$$\bigcap_{z \in D} \overline{F}(z) \neq \emptyset.$$

From the closed version of Theorem 2.1, we deduced the following equivalent formulation in [9]:

**Theorem 2.2.** Let  $(X, D; \Gamma)$  be a  $G$ -convex space and  $F : D \multimap X$  a map such that

- (1)  $\bigcap_{z \in D} \overline{F}(z) = \bigcap_{z \in D} F(z)$  (that is,  $F$  is transfer closed-valued);
- (2)  $\overline{F}$  is a KKM map; and
- (3)  $\bigcap_{z \in M} \overline{F}(z)$  is compact for some  $M \in \langle D \rangle$ .

Then we have  $\bigcap_{z \in D} F(z) \neq \emptyset$ .

For a  $G$ -convex space  $(X \supset D; \Gamma)$ , a subset  $Y$  of  $X$  is called a  $G$ -convex subspace of  $(X \supset D; \Gamma)$  if  $(Y, Y \cap D; \Gamma')$  is a  $G$ -convex space where  $\Gamma'_A := \Gamma_A \cap Y$  for  $A \in \langle Y \cap D \rangle$ .

For a  $G$ -convex space  $(X \supset D; \Gamma)$ , we obtained another form of the KKM theorem with a more general coercivity (or compactness) condition in [9]:

**Theorem 2.3.** Let  $(X \supset D; \Gamma)$  be a  $G$ -convex space,  $K$  a nonempty compact subset of  $X$ , and  $F : D \multimap X$  a multimap such that

- (1)  $\bigcap_{z \in D} F(z) = \bigcap_{z \in D} \overline{F}(z)$ ;
- (2)  $\overline{F}$  is a KKM map; and
- (3) for each  $N \in \langle D \rangle$ , there exists a compact  $G$ -convex subspace  $L_N$  of  $X$  containing  $N$  such that

$$L_N \cap \bigcap \{ \overline{F}(z) : z \in L_N \cap D \} \subset K.$$

Then  $K \cap \bigcap \{ F(z) : z \in D \} \neq \emptyset$ .

### 3. Better admissible multimaps

Let  $(X, D; \Gamma)$  be a  $G$ -convex space and  $Y$  a topological space. We define the better admissible class  $\mathfrak{B}$  of multimaps from  $X$  into  $Y$  as follows [7]:

$F \in \mathfrak{B}(X, Y) \Leftrightarrow F : X \multimap Y$  is a map such that for any  $N \in \langle D \rangle$  with  $|N| = n + 1$  and any continuous map  $p : F(\Gamma_N) \rightarrow \Delta_n$ , the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point.

Let  $X$  be a nonempty convex subset of a t.v.s.  $E$  and  $Y$  a topological space. A polytope  $P$  in  $X$  is any convex hull of a nonempty finite subset of  $X$ ; or a nonempty compact convex subset of  $X$  contained in a finite dimensional subspace of  $E$ .

For multimaps defined on a convex set  $X$ , the better admissible class  $\mathfrak{B}$  reduces to the following [10–12]:

$F \in \mathfrak{B}(X, Y) \Leftrightarrow F: X \multimap Y$  is a map such that for any polytope  $P$  in  $X$  and any continuous map  $f: F(P) \rightarrow P$ ,  $f \circ (F|_P): P \multimap P$  has a fixed point.

Subclasses of  $\mathfrak{B}$  are classes of continuous functions  $\mathbb{C}$ , the Kakutani maps  $\mathbb{K}$  (u.s.c. with closed convex values and codomains are convex spaces), the Aronszajn maps  $\mathbb{M}$  (u.s.c. with  $R_\delta$  values), the acyclic maps  $\mathbb{V}$  (u.s.c. with acyclic values), the Powers maps  $\mathbb{V}_c$  (finite compositions of acyclic maps), the O’Neill maps  $\mathbb{N}$  (continuous with values of one or  $m$  acyclic components, where  $m$  is fixed), the approachable maps  $\mathbb{A}$  (whose domains and codomains are uniform spaces), admissible maps of Górniewicz,  $\sigma$ -selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class  $\mathbb{K}_c^+$  of Lassonde, the class  $\mathbb{V}_c^+$  of Park et al., and approximable maps of Ben-El-Mechaiekh and Idzik, and many others. Those subclasses are all examples of the admissible class  $\mathfrak{A}_c^k$  of Park. Some examples of maps in  $\mathfrak{B}$  not belonging to  $\mathfrak{A}_c^k$  were given recently. For details, see [7,10–12].

We give another class of multimaps:

For any topological space  $Y$  and a  $G$ -convex space  $(X, D; \Gamma)$ , a map  $T: Y \multimap X$  is called a  $\Phi$ -map if there exists a map  $S: Y \multimap D$  such that

- (i) for each  $y \in Y$ ,  $M \in \langle S(y) \rangle$  implies  $\Gamma_M \subset T(y)$ ; and
- (ii)  $Y = \bigcup \{\text{int } S^-(x) : x \in D\}$ .

The following is given in [14]:

**Lemma 3.1.** *Let  $K$  be a Hausdorff compact space,  $(X, D; \Gamma)$  a  $G$ -convex space, and  $T: K \multimap X$  a  $\Phi$ -map. Then  $T$  has a continuous selection  $f: K \rightarrow X$ ; that is,  $f(y) \in T(y)$  for all  $y \in K$ . More precisely, there exist two continuous functions  $p: K \rightarrow \Delta_n$  and  $\phi_N: \Delta_n \rightarrow \Gamma_N$  such that  $f = \phi \circ p$  for some  $N \in \langle D \rangle$  with  $|N| = n + 1$ .*

#### 4. Existence of equilibria

In this section, we begin with the following general existence theorem for solutions of vectorial equilibrium problems involving  $g$ -monotone multimaps:

**Theorem 4.1.** *Let  $(X \supset Y; \Gamma)$  be a  $G$ -convex space and  $Z$  a nonempty set. Let  $C: X \multimap Z$ ,  $D: Y \multimap Z$ ,  $f: X \times X \multimap Z$ ,  $g: X \times Y \multimap Z$  be multimaps and  $F: Y \multimap X$  a multimap defined by  $F(y) := \{x \in X: g(x, y) \subset D(y)\}$  for  $y \in Y$ . Suppose that*

- (i)  $\forall x \in X, \forall y \in Y, f(x, y) \not\subset C(x) \Rightarrow g(x, y) \subset D(y)$ ;

- (ii)  $F$  is transfer closed-valued;
- (iii)  $\forall x \in X, \{y \in X: f(x, y) \subset C(x)\}$  is  $\Gamma$ -convex;
- (iv)  $\forall x \in X, g(x, y) \subset D(y) \forall y \in Y \Rightarrow f(x, y) \not\subset C(x) \forall y \in Y$ ;
- (v)  $\forall x \in X, f(x, x) \not\subset C(x)$ ; and
- (vi)  $\bigcap_{y \in M} \overline{F}(y)$  is compact for some  $M \in \langle Y \rangle$ .

Then there exists an  $\bar{x} \in X$  such that  $f(\bar{x}, y) \not\subset C(\bar{x})$  for all  $y \in Y$ .

**Proof.** Note that (ii) means

$$\bigcap_{y \in Y} F(y) = \bigcap_{y \in Y} \overline{F}(y). \tag{6}$$

We claim that  $F: Y \multimap X$  is a KKM map. In fact, for any  $\{y_1, \dots, y_n\} \in \langle Y \rangle$ , suppose that there exists an  $x \in \Gamma(\{y_1, \dots, y_n\})$  such that  $x \notin F(y_i)$  for all  $i$ . Then  $g(x, y_i) \not\subset D(y_i)$  for all  $i = 1, 2, \dots, n$  and hence, by (i), we have  $f(x, y_i) \subset C(x)$ . Now by (iii), we have  $f(x, x) \subset C(x)$ , which contradicts (v). Therefore,

$$\overline{F}: Y \multimap X \text{ is a KKM map.} \tag{7}$$

Note that (vi) is the same to condition (3) of Theorem 2.2 with  $Y = D$ . Therefore, by Theorem 2.2, we have an  $\bar{x} \in \bigcap_{y \in Y} F(y)$ . Hence  $g(\bar{x}, y) \subset D(y)$  for all  $y \in Y$ . By (iv), this implies  $f(\bar{x}, y) \not\subset C(\bar{x})$  for all  $y \in Y$ .  $\square$

**Remark.** Theorem 4.1 reduces to Oettli and Schläger [5, Theorem 1] whenever  $X = Y$  is a compact convex subset of a t.v.s.,  $Z$  is a t.v.s., and  $F$  is closed-valued.

**Theorem 4.1'.** Under the hypothesis of Theorem 4.1, let us replace (vi) by the following:

- (vi') there exists a nonempty compact subset  $K$  of  $X$  such that for any  $N \in \langle Y \rangle$ , there exists a compact  $G$ -convex subspace  $L_N$  of  $X$  containing  $N$  such that

$$L_N \cap \bigcap_{y \in L_N \cap Y} \overline{F}(y) \subset K.$$

Then there exists an  $\bar{x} \in K$  such that  $f(\bar{x}, y) \not\subset C(\bar{x})$  for all  $y \in Y$ .

**Proof.** As in the proof of Theorem 4.1,  $\overline{F}: Y \multimap X$  is a KKM map. Further, (vi') implies condition (3) of Theorem 2.3. Therefore, by our KKM Theorem 2.3, we have an  $\bar{x} \in K$  such that  $\bar{x} \in \bigcap \{F(y): y \in Y\}$ . Hence  $g(\bar{x}, y) \subset D(y)$  for all  $y \in Y$ . By (iv), this implies  $f(\bar{x}, y) \not\subset C(\bar{x})$  for all  $y \in Y$ . This completes our proof.  $\square$

**Remark.** In the case when  $X = Y$  is a convex subset of a t.v.s.,  $Z$  a t.v.s., and  $F$  is closed-valued, Oettli and Schläger [5, Remark 1] observed that the following coercivity condition works instead of (vi):

(vi'') *there exists a nonempty compact set  $K \subset X$  and a compact convex subset  $L \subset X$  such that, for every  $x \in X \setminus K$ , there exists  $y \in L$  with  $g(x, y) \not\subset D(y)$ .*

We show that (vi'')  $\Rightarrow$  (vi'). In fact, we can put  $L_N = \overline{\text{co}}(L \cup N)$ . For any  $x \in X \setminus K$  and  $y \in L \subset L_N$  satisfying  $g(x, y) \not\subset D(y)$ , we have

$$x \notin \{x \in X: g(x, y) \subset D(y)\} = \overline{F}(y).$$

Let  $X$  and  $Z$  be t.v.s. According to [5], given a multimap  $h: X \multimap Z$  and a convex cone  $P \subset Z$ , we say that  $h$  is *right  $P$ -convex* if, for all  $x, y \in X$  and  $\alpha \in [0, 1]$ ,

$$h(\alpha x + (1 - \alpha)y) \subset \alpha h(x) + (1 - \alpha)h(y) - P.$$

Similarly,  $h$  is said to be *left  $P$ -convex* if

$$h(\alpha x + (1 - \alpha)y) + P \supset \alpha h(x) + (1 - \alpha)h(y).$$

If  $h$  is single-valued, then both notions coincide with ordinary  $P$ -convexity.

Let  $X$  be a nonempty convex subset of a t.v.s.  $E$ . For every  $x \in X$ , let  $P(x)$  be a closed convex cone (not necessarily pointed) of a t.v.s.  $Z$  such that  $\text{int } P(x) \neq \emptyset$  and  $P(x) \neq Z$ .

We use Theorem 4.1' to the case  $X = Y$ ,  $g(x, y) := -f(x, y)$ ,  $C(x) := -\text{int } P(x)$ ,  $D(x) := -P(x)$ , and obtain the following more practical form as in [5]:

**Corollary 4.2.** *Let  $X$  be a nonempty convex subset of a t.v.s.  $E$  and  $f: X \times X \multimap Z$  a map such that*

- (i) *for all  $x, y \in X$ ,  $f(x, y) \not\subset -\text{int } P(x)$  implies  $f(y, x) \subset -P(y)$ ;*
- (ii) *for all  $y \in X$ ,  $f(y, \cdot)$  is l.s.c.;*
- (iii) *for all  $x \in X$ ,  $f(x, \cdot)$  is right  $P(x)$ -convex;*
- (iv) *the map  $\text{int } P(\cdot)$  has open graph in  $X \times Z$ ;*
- (v) *for all  $x, y \in X$ ,  $f(\cdot, y)$  is u.s.c. and compact-valued on  $[x, y]$ ;*
- (vi) *for all  $x \in X$ ,  $f(x, x) \not\subset -\text{int } P(x)$ ; and*
- (vii) *there exists a nonempty compact subset  $K$  of  $X$  such that for any  $N \in \langle X \rangle$ , there exists a compact convex subset  $L_N$  of  $X$  containing  $N$  such that*

$$L_N \setminus K \subset \bigcup_{y \in L_N} \{x \in X: f(x, y) \subset -\text{int } P(x)\}.$$

Then there exists

$$\bar{x} \in K \quad \text{such that} \quad f(\bar{x}, y) \not\subset -\text{int } P(\bar{x}) \quad \forall y \in X.$$

**Remark.** [5, Corollary 1] is just Corollary 4.2 with condition (vi'') instead of (vii).

### 5. Mixed equilibria

In this section, let  $E$  and  $Z$  be t.v.s. and  $L(E, Z)$  denote the space of all continuous linear operators  $E \rightarrow Z$ . For  $\phi \in L(E, Z)$ , we write  $\langle \phi, x \rangle := \phi(x)$  and, for  $\Phi \subset L(E, Z)$ ,  $\langle \Phi, x \rangle := \{\langle \phi, x \rangle : \phi \in \Phi\}$ . Suppose that  $L(E, Z)$  is topologized in such a way that it is a Hausdorff t.v.s. and  $\langle \cdot, \cdot \rangle$  is continuous on  $M \times E$  whenever  $M \subset L(E, Z)$  is compact. The map  $P : X \rightarrow Z$  remains as in Section 4.

**Theorem 5.1.** *Let  $X$  be a nonempty compact convex subset of  $E$ ,  $M \subset L(E, Z)$  a nonempty compact convex subset, and  $f, g : X \times X \rightarrow Z$ . Suppose that*

- (i)  $T \in \mathfrak{B}(X, M)$ ;
- (ii) for all  $y \in X$ ,  $g(\cdot, y)$  is l.s.c. and compact-valued;
- (iii) for all  $x \in X$ ,  $f(x, \cdot)$  and  $g(\cdot, x)$  are right  $P(x)$ -convex;
- (iv) for all  $x, y \in X$  and  $\xi \in M$ , if  $f(x, y) + \langle \xi, y - x \rangle \not\subset -\text{int } P(x)$ , then  $g(x, y) - \langle \xi, y - x \rangle \subset -P(y)$ ;
- (v) for all  $x \in X$  and  $\xi \in M$ , if  $g(x, y) - \langle \xi, y - x \rangle \subset -P(y) \quad \forall y \in X$ , then  $f(x, y) + \langle \xi, y - x \rangle \not\subset -\text{int } P(x) \quad \forall y \in X$ ; and
- (vi) for all  $x \in X$ ,  $f(x, x) \not\subset -\text{int } P(x)$ .

Then there exist  $\bar{x} \in X$  and  $\bar{\xi} \in T(\bar{x})$  such that

$$f(\bar{x}, y) + \langle \bar{\xi}, y - \bar{x} \rangle \not\subset -\text{int } P(\bar{x}) \quad \forall y \in X.$$

**Proof.** By condition (v) it suffices to show the existence of  $(\bar{x}, \bar{\xi})$  satisfying

$$\bar{x} \in X, \quad \bar{\xi} \in T(\bar{x}), \quad g(\bar{x}, y) - \langle \bar{\xi}, y - \bar{x} \rangle \subset -P(y) \quad \forall y \in X. \quad (8)$$

Since  $-P(y) = \bigcap \{z - P(y) : z \in \text{int } P(y)\}$ , (8) holds if, for all  $y \in X$  and  $z \in \text{int } P(y)$ ,  $(\bar{x}, \bar{\xi})$  is contained in

$$R(y, z) := \{(x, \xi) \in X \times M : \xi \in T(x), g(x, y) - \langle \xi, y - x \rangle \subset z - P(y)\}.$$

As in the proof of [5, Theorem 2], the Hausdorff compact set  $M$  is covered by the open sets

$$U(x) := \bigcap_{j=1}^m \{\xi \in M : g(x, y_j) - \langle \xi, y_j - x \rangle \subset z_j - \text{int } P(y_j)\}, \quad x \in X.$$

Hence there exist a finite subcover  $U(x_1), \dots, U(x_n)$  and a continuous partition of unity  $\beta_1, \dots, \beta_n$  subordinate to this subcover; i.e., nonnegative continuous functions  $\beta_i : M \rightarrow \mathbb{R}$  with  $\sum_{i=1}^n \beta_i(\xi) = 1$  for all  $\xi \in M$  and  $\xi \in U(x_i)$  whenever  $\beta_i(\xi) \neq 0$ . Then  $p(\xi) := \sum_{i=1}^n \beta_i(\xi)x_i \in Q = \text{co}\{x_i : 1 \leq i \leq n\} \subset X$  defines a continuous map  $p : M \rightarrow X$ . Since  $T \in \mathfrak{B}(X, M)$ ,  $(p|_{T(Q)})(T|_Q) : Q \rightarrow Q$  has a fixed point  $x' \in p(T(x'))$ . Hence there exists  $\xi' \in T(x')$  such that  $x' := p(\xi') = \sum_{i \in I} \beta_i(\xi')x_i$ , where  $I := \{i : \beta_i(\xi') > 0\}$ . Then, from (iii) and  $\xi' \in U(x_i)$  for all  $i \in I$ , it follows for all  $j$  that

$$\begin{aligned} g(x', y_j) - \langle \xi', y_j - x' \rangle &\subset \sum_{i \in I} \beta_i(\xi') (g(x_i, y_j) - \langle \xi', y_j - x_i \rangle) - P(y_j) \\ &\subset z_j - \text{int } P(y_j) - P(y_j) \\ &\subset z_j - P(y_j), \end{aligned}$$

hence  $(x', \xi') \in R(y_j, z_j)$ , and this proves (8).  $\square$

**Remarks.** 1. [5, Theorem 2] is a particular case of Theorem 5.1 when  $L(E, Z)$  is locally convex and  $T : X \rightarrow M$  is u.s.c. with nonempty closed convex values.

2. Theorem 5.1 remains true whenever the compactness of  $X$  is replaced by the following coercivity condition:

- (vii) *there exists a nonempty compact subset  $K$  of  $X$  such that for each  $N \in \langle X \rangle$  there exists a compact convex subset  $L_N$  of  $X$  containing  $N$  such that for each  $x \in L_N \setminus K$  and  $\xi \in T(x)$  there exists a  $y \in L_N$  with  $g(x, y) - \langle \xi, y - x \rangle \not\subset -P(y)$ .*

The proof is similar to that of remark after Theorem 4.1'. As in [5], from Theorem 5.1, we can deduce the following:

**Corollary 5.2.** *Let  $X$  be a nonempty convex subset of  $E$ ,  $M \subset L(E, Z)$  a non-empty compact convex subset, and  $f : X \times X \rightarrow Z$ . Suppose that*

- (i)  $T \in \mathfrak{B}(X, M)$ ;
- (ii) for all  $y \in X$ ,  $f(y, \cdot)$  is l.s.c. and compact-valued;
- (iii) for all  $x \in X$ ,  $f(x, \cdot)$  is right  $P(x)$ -convex;
- (iv) for all  $x, y \in X$  and  $\xi \in M$ , if  $f(x, y) + \langle \xi, y - x \rangle \not\subset -\text{int } P(x)$ , then  $f(y, x) - \langle \xi, y - x \rangle \subset -P(y)$ ;
- (v) for all  $x, y \in X$ ,  $f(\cdot, y)$  is u.s.c. on  $[x, y]$ , and  $\text{int } P(\cdot)$  has open graph;
- (vi) for all  $x \in X$ ,  $f(x, x) \not\subset -\text{int } P(x)$ ; and
- (vii) *there exists a nonempty compact subset  $K$  of  $X$  such that for each  $N \in \langle X \rangle$  there exists a compact convex subset  $L_N$  of  $X$  containing  $N$  such that for each  $x \in L_N \setminus K$  and  $\xi \in T(x)$  there exists a  $y \in L_N$  with  $f(x, y) + \langle \xi, y - x \rangle \subset -\text{int } P(x)$ .*

Then there exist an  $\bar{x} \in K$  and a  $\bar{\xi} \in T(\bar{x})$  such that

$$f(\bar{x}, y) + \langle \bar{\xi}, y - \bar{x} \rangle \not\subset -\text{int } P(\bar{x}) \quad \forall y \in X.$$

**Remark.** If  $L(E, Z)$  is locally convex and  $T$  is u.s.c. with nonempty closed convex values, then Corollary 5.2 reduces to [5, Corollary 2].

### 6. Further existence results

In this section, we obtain generalized forms of the results of Section 4 in [5]:

**Theorem 6.1.** Let  $A$  be a Hausdorff space,  $(B \supset D; \Gamma)$  a  $G$ -convex space, and  $T \in \mathfrak{B}(B, A)$  a compact map. Let  $G, H \subset A \times B$  have the following properties:

- (i) for all  $y \in B$  and  $x \in T(y)$ ,  $(x, y) \in H$ ;
- (ii) for all  $y \in B$ ,  $\{x \in A: (x, y) \notin G\}$  is open in  $A$ ;
- (iii) for all  $x \in A$ ,  $\{x \in B: (x, y) \notin H\}$  is  $\Gamma$ -convex; and
- (iv)  $H \subset G$ .

Then there exists an  $\bar{x} \in A$  such that  $(\bar{x}, y) \in G$  for all  $y \in B$ .

**Proof.** Suppose that such an  $\bar{x}$  does not exist. Then for each  $x \in A$ , there exists a  $y \in B$  such that  $(x, y) \notin G$  and hence  $(x, y) \notin H$  by (iv). Let  $K := \overline{T(B)}$  and define a multimap  $S: K \multimap B$  by

$$S(x) := \{y \in B: (x, y) \notin H\}, \quad x \in K.$$

Then each  $S(x)$  is  $\Gamma$ -convex by (iii), and

$$\begin{aligned} K \subset A &= \bigcup_{y \in B} \{x \in A: (x, y) \notin G\} = \bigcup_{y \in B} \text{int}\{x \in A: (x, y) \notin H\} \\ &= \bigcup_{y \in B} \text{int } S^-(y). \end{aligned}$$

Since  $K$  is compact and  $S: K \multimap B$  is a  $\Phi$ -map, by Lemma 3.1,  $S$  has a continuous selection  $f: K \rightarrow B$  such that  $f = \phi_N \circ p$  for some  $p: K \rightarrow \Delta_n$ ,  $\phi_N: \Delta_n \rightarrow \Gamma_N$ , and  $N \in \langle D \rangle$ . Since  $T \in \mathfrak{B}(B, A)$ , the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{T|_{\Gamma_N}} T(\Gamma_N) \hookrightarrow K \xrightarrow{p} \Delta_n$$

has a fixed point  $e_0 \in \Delta_n$ . Then  $e_0 \in (p \circ T \circ \phi_N)(e_0)$  and

$$y_0 := \phi_N(e_0) \in (\phi_N \circ p \circ T)(y_0) = (f \circ T)(y_0) \subset (S \circ T)(y_0).$$

Hence there exists an  $x_0 \in T(y_0)$  such that  $y_0 \in S(x_0)$ . Note that  $x_0 \in T(y_0)$  implies  $(x_0, y_0) \in H$  by (i). On the other hand,  $y_0 \in S(x_0)$  implies  $(x_0, y_0) \notin H$ , a contradiction.  $\square$

For a convex subset  $B$  of a t.v.s., Theorem 6.1 reduces to the following:

**Theorem 6.2.** *Let  $A$  be a Hausdorff space,  $B$  a convex subset of a t.v.s.,  $G, H \subset A \times B$ , and  $T \in \mathfrak{B}(B, A)$  a compact map. Suppose that (i)–(iv) of Theorem 6.1 holds (obviously  $\Gamma$ -convex means the usual convexity). Then there exists an  $\bar{x} \in A$  such that  $(\bar{x}, y) \in G$  for all  $y \in B$ .*

**Remark.** [5, Theorem 3] is a particular form of Theorem 6.2 for the case where  $A$  is a compact convex subset of a locally convex t.v.s. and  $T$  is an u.s.c. multimap with nonempty closed convex values.

From Theorem 6.2, we obtain the following improved version of [5, Theorem 4]:

**Theorem 6.3.** *Let  $A$  be a Hausdorff compact space,  $B$  a convex subset of a t.v.s., and  $T \in \mathfrak{B}(B, A)$  such that  $T(B) = A$ . Let  $F, G \subset A \times B$  have the following properties:*

- (i) for all  $y \in B$  and  $x \in T(y)$ ,  $(x, y) \in F$ ;
- (ii) for all  $y \in B$ ,  $\{x \in A: (x, y) \notin G\}$  is open in  $A$ ;
- (iii) for all  $x \in A$ ,  $\{y \in B: (x, y) \notin F\}$  is convex;
- (iv)  $F \subset G$ ; and
- (v) for all  $v, y \in B$ ,  $v \neq y$ , and all  $x \in T(v)$ ,

$$(x, u) \in G \quad \forall u \in [v, y] \quad \text{implies} \quad (x, y) \in F.$$

Then there exists  $\bar{x} \in A$  such that  $(\bar{x}, y) \in F$  for all  $y \in B$ .

**Proof.** By putting  $H := F$  in Theorem 6.2, we have an  $\bar{x} \in A$  such that  $(\bar{x}, u) \in G$  for all  $u \in B$ . Since  $\bar{x} \in T(v)$  for some  $v \in B$ , it follows from (v) and (i) that  $(\bar{x}, y) \in F$  for all  $y \in B$ .  $\square$

Literature on examples can be seen in [5] and references therein.

## References

- [1] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123–145.

- [2] M. Bianchi, N. Hadjisavvas, S. Schaible, Vector equilibrium problems with generalized monotone bifunctions, *J. Optim. Theory Appl.* 92 (1997) 531–546.
- [3] Q.H. Ansari, W. Oettli, D. Schläger, A generalization of vectorial equilibria, *Math. Methods Oper. Res.* 46 (1997) 147–152.
- [4] W. Oettli, D. Schläger, Generalized vectorial equilibria and generalized monotonicity, in: M. Brokate, A.H. Siddigi (Eds.), *Functional Analysis with Current Applications*, Longman, London, 1998, pp. 145–154.
- [5] W. Oettli, D. Schläger, Existence of equilibria for  $g$ -monotone mappings, in: W. Takahashi, T. Tanaka (Eds.), *Nonlinear Analysis and Convex Analysis*, World Scientific, Singapore, 1999, pp. 26–33.
- [6] S. Park, Elements of the KKM theory for generalized convex spaces, *Korean J. Comput. Appl. Math.* 7 (2000) 1–28.
- [7] S. Park, Fixed points of better admissible maps on generalized convex spaces, *J. Korean Math. Soc.* 37 (2000).
- [8] S. Park, Fixed point theorems in locally  $G$ -convex spaces, *Nonlinear Anal.*
- [9] S. Park, Remarks on topologies of generalized convex spaces, *Nonlinear Funct. Anal. Appl.* 5 (2000) 67–79.
- [10] S. Park, Coincidence theorems for the better admissible multimaps and their applications, *Nonlinear Anal.* 30 (1997) 4183–4191.
- [11] S. Park, Fixed points of the better admissible multimaps, *Math. Sci. Res. Hot-Line* 1 (1997) 1–6.
- [12] S. Park, A unified fixed point theory of multimaps on topological vector spaces, *J. Korean Math. Soc.* 35 (1998) 803–829, corrections: 36 (1999) 829–832.
- [13] S. Park, B. Sims, Remarks on fixed point theorems on hyperconvex spaces, *Nonlinear Funct. Anal. Appl.* 5 (2000) 51–64.
- [14] S. Park, Continuous selection theorems in generalized convex spaces, *Numer. Funct. Anal. Optim.* 20 (1999) 567–583.