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Remarks on Acyclic Versions of Generalized von Neumann and Nash Equilibrium Theorems

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Abstract—A fixed-point theorem on compact compositions of acyclic maps on admissible (in the sense of Klee) convex subset of a t.v.s. is applied to obtain a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem. Our new results generalize earlier works of Lassonde [1], Simons [2], and Park [3,4]. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Kakutani map, Acyclic map, Admissible subset (in the sense of Klee), Coincidence, Intersection theorem, Equilibrium theorem, Saddle point, Minimax theorem.

1. INTODUCTION

Von Neumann's 1928 minimax theorem [5] and 1937 intersection lemma [6] have numerous generalizations and applications. Kakutani's 1941 fixed-point theorem [7] was to give simple proofs of the above-mentioned results. Nash [8] obtained his 1951 equilibrium theorem based on the Brouwer or Kakutani fixed-point theorem. In 1952, Fan [9] and Glicksberg [10] extended the Kakutani theorem to locally convex Hausdorff topological vector spaces. This result was applied by its authors to generalize the von Neumann intersection lemma and the Nash equilibrium theorem. Further generalizations were followed by Ma [11] and others. For the literature, see [12] and references therein.

An upper semicontinuous (u.s.c.) multimap with nonempty compact convex values is called a *Kakutani map*. The Fan-Glicksberg theorem was extended by Himmelberg [13] in 1972 for compact Kakutani maps instead of assuming compactness of domains. In 1990, Lassonde [1] extended the Himmelberg theorem to multimaps factorizable by Kakutani maps through convex sets in Hausdorff topological vector spaces. Moreover, Lassonde applied his theorem to game theory and obtained a von Neumann type intersection theorem for finite number of sets and a Nash type equilibrium theorem comparable to Debreu's social equilibrium existence theorem [14].

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Recall that a nonempty topological space is said to be *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any convex or star shaped subset of a topological vector space is contractible, and any contractible space is acyclic. A multimap is said to be *acyclic* if it is u.s.c. with compact acyclic values. In 1946, the Kakutani fixed-point theorem was extended for acyclic maps by Eilenberg and Montgomery [15]. This result was applied by the present author [16] to give acyclic versions of the social equilibrium existence theorem due to Debreu [14], saddle point theorems, minimax theorems, and the Nash equilibrium theorem. Moreover, the present author [17,18] obtained a fixed-point theorem for compact compositions of acyclic maps defined on admissible (in the sense of Klee) convex subsets of topological vector spaces. This new fixed-point theorem was applied in [4] to deduce acyclic versions of the von Neumann intersection theorem, the minimax theorem, the Nash equilibrium theorem, and others.

Our aim in this paper is to show that, by following Lassonde's method in [1], our fixed-point theorem implies more general formulations of results of Lassonde [1], Park [3,4], and others.

In Section 3, we give a Simons type cyclic coincidence theorem for acyclic maps as an application of our fixed-point theorem. Section 4 deals with the von Neumann type intersection theorem for graphs of compact compositions of acyclic maps. Finally, in Section 5, we obtain the Nash type equilibrium theorem under much general assumptions. A saddle point or minimax theorem is added.

2. PRELIMINARIES

A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^-(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a map $T : X \multimap Y$ is sad to be *closed* if its graph $\mathrm{Gr}(T) = \{(x,y) : x \in X, y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. map T with closed values is closed.

Let $\mathbb{K}$ denote the class of the Kakutani maps, $\mathbb{K}_c$ finite compositions of the Kakutani maps, $\mathbb{V}$ the class of acyclic maps, and $\mathbb{V}_c$ finite compositions of acyclic maps. For example, $\mathbb{V}_c(X,Y) = \{F : X \multimap Y \mid F \in \mathbb{V}_c\}$.

Throughout this paper, all topological vector spaces (t.v.s.) are assumed to be Hausdorff.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite-dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. is admissible and that every compact convex locally convex subset of a t.v.s. is admissible. For details, see [18] and references therein.

The following particular form of our fixed-point theorem in [17,18] is the basis of our arguments in this paper.

THEOREM 1. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $F \in \mathbb{V}_c(X,X)$ has a fixed point.*

Let $\mathbb{Z}_{n+1} = \{0, 1, \ldots, n\}$ with $n+1$ interpreted as 0.

3. CYCLIC COINCIDENCE THEOREMS FOR ACYCLIC MAPS

Simons [2] obtained several interesting results on cyclical coincidences and Park [3] generalized all the main results of [2] to much wider classes of multimaps, for example, to acyclic maps instead of Kakutani maps.

In this section, as an application of Theorem 1, we give a generalization of only one of the results in [3] and note that the other results in it may have similar generalized forms.

THEOREM 2. *Let $n \geq 0$ and, for each $i \in \mathbb{Z}_{n+1}$, let Y_i be a convex set in a t.v.s. E_i, and $V_i \in \mathbb{V}(Y_k, Y_{k+1})$ be compact. If $Y_0 \times Y_1 \times \cdots \times Y_n$ is an admissible subset of $E_0 \times E_1 \times \cdots \times E_n$, then there exists $(y_0, y_1, \dots, y_n) \in Y_0 \times Y_1 \times \cdots \times Y_n$ such that $y_{i+1} \in V_i(y_i)$ for all $i \in \mathbb{Z}_{n+1}$.*

PROOF.

CASE 1. $n = 0$. This follows from Theorem 1.

CASE 2. $n \geq 1$. Let $X = Y_0 \times \cdots \times Y_n$ and $E = E_0 \times \cdots \times E_n$ and define $V : X \multimap X$ by

$$V(y_0, \dots, y_n) = V_n(y_n) \times V_0(y_0) \times \cdots \times V_{n-1}(y_{n-1})$$

for $(y_0, \dots, y_n) \in X$. Note that X is an admissible convex subset of E, $V \in \mathbb{V}(X, X)$, and V is compact. Therefore, by Theorem 1 or Case 1, there exists an $x = (y_0, \dots, y_n) \in X$ such that $x \in V(x)$. This completes our proof.

REMARK 1. Note that X is admissible in E if each Y_i is admissible in E_i.

REMARK 2. If all Y_i are nonempty compact convex subsets of locally convex t.v.s., Theorem 2 reduces to our previous work [3, Theorem 3], which generalizes earlier works of Simons [2, Theorem 2.5] and a number of others; see [3].

4. THE VON NEUMANN TYPE INTERSECTION THEOREM

Let $\{X_i\}_{i \in I}$ be a family of nonempty sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j \in I} X_j \qquad \text{and} \qquad X^i = \prod_{j \in I \setminus \{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the j^{th} coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its i^{th} coordinate is x_i and, for $j \neq i$, the j^{th} coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x in X^i.

For a subset $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A\left(x^i\right) := \left\{y_i \in X_i : \left[x^i, y_i\right] \in A\right\} \qquad \text{and} \qquad A(x_i) := \left\{y^i \in X^i : \left[y^i, x_i\right] \in A\right\}.$$

In our previous work [4, Theorem 2′], we obtained the following.

THEOREM 3. *Let I be any index set, $\{X_i\}_{i \in I}$ a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of $X = \Pi_{j \in I} X_j$ for each $i \in I$. Suppose that for each $x^i \in X^i$, $A_i(x^i)$ is a nonempty convex subset of K_i except a finite number of i's for which $A_i(x^i)$ is an acyclic subset of K_i. If X is admissible in $E = \Pi_{j \in I} E_j$, then $\bigcap_{j \in I} A_j \neq \emptyset$.*

REMARKS 1. If $I = \{1, 2\}$, E_i are Euclidean, $X_i = K_i$, and $A_i(x^i)$ are nonempty and convex, then Theorem 2 reduces to the intersection lemma of von Neumann [6].

REMARK 2. If each $X_i = K_i$ is a compact convex subset of a locally convex t.v.s. E_i and $A_i(x^i)$ is a nonempty convex subset of X_i for each i, then Theorem 3 reduces to Fan [9, Theorem 2], from which Ma [11] could deduce generalizations of the Fan intersection theorem, Fan's analytic alternative, and the Nash theorem in [9,19] for arbitrary family of sets. Ma's results are generalized by Park [20, Theorems 4.3, 8.1, and 8.2]. Note that Remark (2) of [4, Theorem 2′] is incorrectly stated.

For a finite family, we have the following.

THEOREM 4. *Let X_0 be a topological space, and $X_1, X_2, \dots, X_n$ be $n\ (\geq 1)$ convex sets, each in a t.v.s. E_i for each $i = 1, \dots, n$; and let $F_i \in \mathbb{V}_c(X^i, X_i)$ for each $i = 0, 1, \dots, n$. If X^0 is admissible in E^0 and if all the multimaps F_i are compact except possibly F_n, then $\bigcap_{i=0}^n \mathrm{Gr}(F_i) \neq \emptyset$.*

PROOF. For each $i \in \mathbb{Z}_{n+1}$, define a multimap $V_i : X^i \multimap X^{i+1}$ by letting

$$V_i\left(x^i\right) := F_i\left(x^i\right) \times \prod_{\substack{j \in \mathbb{Z}_{n+1} \\ j \notin \{i, i+1\}}} \left\{x_j^i\right\}, \qquad \text{for } x^i \in X^i.$$

Then $V_i \in \mathbb{V}_c(X^i, X^{i+1})$ for each $i \in \mathbb{Z}_{n+1}$. Hence, the multimap $V : X^0 \to X^0$ defined by $V = V_n \circ V_{n-1} \circ \cdots \circ V_0$ belongs to $\mathbb{V}_c(X^0, X_0)$.

(1) X^0 is an admissible convex subset of the t.v.s. $E^0 = \prod_{i=1}^n E_i$.
(2) We claim that V is compact. In fact, for each $i = 0, 1, \dots, n-1$, let K_i be a compact subset satisfying $F_i(X^i) \subset K_i \subset X_i$.

Note that

$$V_0\left(X^0\right) \subset K_0 \times X_2 \times \cdots \times X_n,$$
$$V_1 \circ V_0\left(X^0\right) \subset K_0 \times K_1 \times X_3 \times \cdots \times X_n,$$

and finally,

$$V_{n-1} \circ V_{n-2} \circ \cdots \circ V_0\left(X^0\right) \subset K_0 \times K_1 \times \cdots \times K_{n-1}.$$

Therefore, $V(X^0)$ is contained in the compact set $V_n(K_0 \times K_1 \times \cdots \times K_{n-1})$. Thus, V is compact.

Therefore, by Theorem 1, $V \in \mathbb{V}_c(X^0, X^0)$ has a fixed point $x^0 \in V(x^0)$. Hence, there exist $x^1 \in X^1, \dots, x^n \in X^n$ such that $x^{i+1} \in V_i(x^i)$ for each $i \in \mathbb{Z}_{n+1}$, which implies

$$x_i^{i+1} \in F_i\left(x^i\right), \qquad \text{for each } i \in \mathbb{Z}_{n+1} \tag{1}$$

and

$$x_j^{i+1} = x_j^i, \qquad \text{for each } j \in \mathbb{Z}_{n+1}, \quad j \neq i, \quad j \neq i+1. \tag{2}$$

From (2), it follows that $x_j^i = x_j^k$ for any $i, j, k \in \mathbb{Z}_{n+1}$ with $j \neq i$, $j \neq k$. Therefore, $[x^i, x_i^{i+1}] = [x^k, x_k^{k+1}]$ for any $i, k \in \mathbb{Z}_{n+1}$. Let us denote by x the point of X given by $x := [x^i, x_i^{i+1}]$ for any $i \in \mathbb{Z}_{n+1}$. From (1), we have $x \in \mathrm{Gr}(F_i)$ for each $i \in \mathbb{Z}_{n+1}$, and hence, $\bigcap_{i=0}^n \mathrm{Gr}(F_i)$ is not empty. This completes our proof.

REMARK 1. If X_0 is a convex set in a t.v.s., $X_1, X_2, \dots, X_n$ are convex subsets of locally convex t.v.s. and $F_i \in \mathbb{K}_c(X^i, X_i)$ for each $i = 0, 1, \dots, n$, then Theorem 4 reduces to Lassonde [1, Theorem 5]. Note that we followed his ingenious proof.

REMARK 2. If X_0 is also a convex subset of a t.v.s., X is admissible in E, and all of the multimaps F_i are compact, then Theorem 4 reduces to [4, Theorem 2].

COROLLARY 4.1. *Let X be a topological space and Y an admissible convex subset of a t.v.s. E. Let $F \in \mathbb{V}_c(X, Y)$ and $G \in \mathbb{V}_c(Y, X)$. If F is compact, then $\mathrm{Gr}(F) \cap \mathrm{Gr}(G) \neq \emptyset$.*

REMARK. Even if X is compact, Corollary 4.1 improves [4, Theorem 3], which in turn generalizes Chang [21, Theorem 3], Fan [9, Theorem 2], and the celebrated intersection lemma of von Neumann [6]. From [4, Theorem 3], we deduced two minimax theorems of the von Neumann type.

5. THE NASH TYPE EQUILIBRIUM THEOREM

From Theorem 4, we have the following generalized form of quasi-equilibrium theorems or social equilibrium existence theorems (in the sense of [14]).

THEOREM 5. *Let X_0 be a topological space and $X_1, X_2, \ldots, X_n$ be $n\,(\geq 1)$ convex subsets, each in a t.v.s. For $i = 0, 1, \ldots, n$, let $S_i : X^i \multimap X_i$ be closed maps with compact values, and $f_i, g_i : X = \prod_{i=0}^{n} X_i \to \mathbf{R}$ u.s.c. real functions. Suppose that for each $i = 0, 1, 2, \ldots, n$,*

(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*
(ii) *the function $M_i : X^i \to \mathbf{R}$ defined by*

$$M_i\left(x^i\right) := \max_{y_i \in S_i(x^i)} g_i\left[x^i, y_i\right], \qquad \text{for } x^i \in X^i$$

is l.s.c.; and

(iii) *for each $x^i \in X^i$, $\{x_i \in S_i(x^i) : f_i[x^i, x_i] \geq M_i(x^i)\}$ is acyclic.*

If X^0 is admissible in E^0 and if all the multimaps S_i are compact except possibly S_n and S_n is u.s.c., then there exists an equilibrium point $\hat{x} \in X$; that is,

$$\hat{x}_i \in S_i\left(\hat{x}^i\right) \quad \text{and} \quad f_i\left(\hat{x}\right) \geq \max_{y_i \in S_i(\hat{x}^i)} g_i\left[\hat{x}^i, y_i\right], \qquad \text{for all } i \in \mathbb{Z}_{n+1}.$$

PROOF. For each $i \in \mathbb{Z}_{n+1}$, define $F_i : X^i \multimap X_i$ by letting

$$F_i\left(x^i\right) := \left\{x_i \in S_i\left(x^i\right) : f_i\left[x^i, x_i\right] \geq M_i\left(x^i\right)\right\}$$

for $x^i \in X^i$. Note that each $F_i(x^i)$ is nonempty by (ii) since $S_i(x^i)$ is compact and $g_i[x^i, \ldots ot]$ is u.s.c. on $S_i(x^i)$. We show that $\mathrm{Gr}(F_i)$ is closed in $X^i \times \overline{S_i(X^i)}$. In fact, let $[x^i_\alpha, y^\alpha_i] \in \mathrm{Gr}(F_i)$ and $[x^i_\alpha, y^\alpha_i] \to [x^i, y_i]$. Then

$$f_i\left[x^i, y_i\right] \geq \overline{\lim_\alpha} f_i\left[x^i_\alpha, y^\alpha_i\right] \geq \overline{\lim_\alpha} M_i\left(x^i_\alpha\right) \geq \underline{\lim_\alpha}\, M_i\left(x^i_\alpha\right) \geq M_i\left(x^i\right)$$

and, since $\mathrm{Gr}(S_i)$ is closed in $X^i \times \overline{S_i(X^i)}$, $y^\alpha_i \in S_i(x^i_\alpha)$ implies $y_i \in S_i(x^i)$. Hence, $[x^i, y_i] \in \mathrm{Gr}(F_i)$. Therefore, all F_i are closed.

Since F_i is compact for all $i \neq n$, F_i is u.s.c. for $i \neq n$. Moreover, S_n is u.s.c. with compact values by assumption and F_n is closed, $F_n = S_n \cap F_n$ is u.s.c. Hence, we have $F_i \in \mathbb{V}(X^i, X_i)$. Therefore, by Theorem 4, we have an $\hat{x} \in \bigcap_{i=1}^{n} \mathrm{Gr}(F_i)$; that is, $\hat{x}_i \in F_i(\hat{x}^i)$ for all $i \in \mathbb{Z}_{n+1}$. this completes our proof.

REMARK 1. In our previous work [16, Theorem 1], we obtained a variation of Theorem 5 for the case X_i are acyclic polyhedra and this generalizes the social equilibrium theorem of Debreu [14].

REMARK 2. Note that Theorem 5 sharpens our earlier results [20, Theorem 5.1] and [4, Theorem 6].

REMARK 3. If S_i are u.s.c., by Berge's theorem, M_i is automatically u.s.c. since g_i is u.s.c. if S_i and g_i are continuous, Condition (ii) holds immediately by Berge's theorem, and hence, each M_i is continuous; see [16].

Therefore, we have the following particular form of Theorem 5.

THEOREM 6. *Let X_0 be a topological space, and $X_1, X_2, \ldots, X_n$ be $n\,(\geq 1)$ convex subsets, each in a t.v.s. For $i = 0, 1, \ldots, n$, let $S_i : X^i \to X_i$ be a continuous multimap with compact values and let $f_i : X = \prod_{i=0}^{n} X_i \to \mathbf{R}$ be a continuous function such that for each i, the following holds:*

- *for each $x^i \in X^i$ and each $\alpha \in \mathbf{R}$, the set $\{x_i \in S_i(x^i) : f_i[x^i, x_i] \geq \alpha\}$ is empty or acyclic.*

If X^0 is admissible and if all the multimaps S_i are compact except possibly S_n, then there is an equilibrium point $\hat{x} \in X$.

REMARK. If X_0 is a convex subset of a t.v.s., each X_i for $i \neq 0$ is a convex subset of a locally convex t.v.s., each S_i is convex-valued, and if for each $x^i \in X^i$, the function $x_i \to f_i[x^i, x_i]$ is quasi-concave on X_i, then Theorem 6 reduces to Lassonde [1, Theorem 6], which in turn extends the Nash equilibrium theorem.

The following is a generalization of the Nash theorem.

COROLLARY 6.1. *Let X_0 be a compact topological space and $X_1, X_2, \dots, X_n$ be $n\,(\geq 1)$ compact convex subsets, each in a t.v.s. For each $i = 0, 1, \dots, n$, let $f_i : X \to \mathbf{R}$ be a continuous function such that*

(1) *for each $x^i \in X^i$ and each $\alpha \in \mathbf{R}$, the set*

$$\left\{x_i \in X_i : f_i\left[x^i, x_i\right] \geq \alpha\right\}$$

is empty or acyclic.

If X^0 is admissible, then there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i\left[\hat{x}^i, y_i\right], \qquad \text{for all } i \in \mathbb{Z}_{n+1}.$$

REMARK 1. A particular form of Corollary 6.1 was given by Park [4, Theorem 7].

REMARK 2. If all X_i are compact convex subsets of Euclidean spaces and if $x_i \mapsto f_i[x^i, x_i]$ is quasi-concave for each $x^i \in X^i$, then Corollary 6.1 reduces to the Nash equilibrium theorem [8].

COROLLARY 6.2. *Let X be a compact topological space and Y an admissible compact convex subset of a t.v.s. Let $f : X \times Y \to \mathbf{R}$ be a continuous function. Suppose that for each $x_0 \in X$ and $y_0 \in Y$, the sets*

$$\left\{x \in X : f(x, y_0) = \max_{\xi \in X} f(\xi, y_0)\right\}$$

and

$$\left\{y \in Y : f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\right\}$$

are acyclic. Then

(1) *f has a saddle point $(X_0, y_0) \in X \times Y$; that is,*

$$\min_{\eta \in Y} f(x_0, y) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0);$$

(2) *we have the minimax equality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

PROOF. Note that a saddle point is a particular case of an equilibrium point for two agents ($i = 0, 1$) in Corollary 6.1 for $X_0 = X$, $X_1 = Y$ and $f_0(x, y) = f(x, y)$, $f_1(x, y) = -f(x, y)$. Then we can follow the proof of Park [4, Corollaries 1 and 2] to obtain the conclusion.

REMARK 1. Corollary 6.2 shows that the von Neumann minimax theorem holds under much milder restriction.

REMARK 2. Corollary 6.2 was obtained by the author [4, Theorems 4 and 5] using Corollary 4.1 under the restriction that X is Hausdorff.

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