ON ALMOST FIXED POINT PROPERTY OF MAPS IN TOPOLOGICAL VECTOR SPACES

Sehie Park

Abstract. We define the (convexly) almost fixed point property on multimaps defined on subsets of topological vector spaces, and obtain typical results related to such property. Simultaneously, we introduce some related fixed point theorems recently obtained by the author in his forthcoming works [21-24].

1. Definitions and basic results

In this paper all spaces are assumed to be Hausdorff and a t.v.s. means a topological vector space. A multimap (simply, a map) $T : X \multimap Y$ is a function from X into $2^Y \setminus \{\emptyset\}$.

Let X be a subset of a t.v.s. E. A multimap $T : X \multimap E$ is said to have *the* (*convexly*) *almost fixed point property* if every (resp. convex) neighborhood U of the origin 0 of E, there exists a point $x_U \in X$ such that

$$x_U \in T(x_U) + U \quad \text{or} \quad T(x_U) \cap (x_U + U) \neq \emptyset.$$

The almost fixed point property is abbreviated to a.f.p.p. and the convexly a.f.p.p. to c.a.f.p.p.

Recall that, for topological spaces X and Y, a multimap $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T)$ is closed in $X \times Y$, and *compact* if its range $T(X)$ is contained in a compact subset of Y.

2000 Mathematics Subject Classification. 47H10, 54C60, 54H25, 55M20.

Key Words and Phrases. (convexly) almost fixed point property, closed multimap, compact multimap, almost convex, the Zima type, admissible (in the sense of Klee), better admissible class of multimaps.

It is well known that any upper semicontinuous map is closed if it has closed values, and has compact range if it has compact domain and compact values.

The following basic principle is routine :

Theorem 1.1. *Let X be a subset of a t.v.s. E, and $T : X \multimap X$ a closed compact map. Then the following are equivalent:*

(i) *T has a fixed point.*

(ii) *T has the almost fixed point property.*

Proof. (i) $\Longrightarrow$ (ii) Clear.

(ii) $\Longrightarrow$ (i) For each neighborhood U of 0, there exist $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(X)$ is relatively compact, we can choose a subnet of the net $\{y_U\}$ with a cluster point $x_0 \in \overline{T(X)}$. Since E is Hausdorff, the corresponding subnet of $\{x_U\}$ also has the cluster point x_0. Because the graph of T is closed in $X \times \overline{T(X)}$, we have $x_0 \in T(x_0)$. This completes our proof.

Theorem 1.2. *Let X be a subset of a locally convex t.v.s. E, and $T : X \multimap X$ a closed compact map. Then the following are equivalent:*

(i) *T has a fixed point.*

(ii) *T has the convexly almost fixed point property.*

Proof. In a locally convex t.v.s., the c.a.f.p.p. is equivalent to the a.f.p.p.

Remarks 1. If T is not compact, then (ii) $\not\Rightarrow$ (i). For example, let $X = \mathbb{R}$ and $T(x) := \{x - 1/x\}$ if $x \neq 0$, $T(0) := \{1, -1\}$.

2. If T is not closed, then (ii) $\not\Rightarrow$ (i). For example, let $X = [0, 1]$ and $T(x) := \{1/2\}$ if $x \neq 1/2$, $T(1/2) := \{0, 1\}$.

The following is recently obtained by the author [23, Theorem 2.5]:

Theorem 1.3. *Let X be a compact convex subset of a t.v.s. E, $\mathcal{V}$ a local base of open neighborhoods of 0 in E, and $T : X \multimap E$ a multimap such that*

(1) *T has the convexly almost fixed point property;*

(2) *T has closed values; and*

(3) *the following equality holds.*

$$(*) \qquad \bigcap_{U \in \mathcal{V}} \{x \in X : x \in T(x) + U\} = \bigcap_{U \in \mathcal{V}} \mathrm{cl}\{x \in X : x \in T(x) + \mathrm{co}\, U\}.$$

Then T has a fixed point.

In view of the Basic Theorems 1.1 - 1.3, if we have a mapping class having the a.f.p.p.(with condition $(*)$), then we can obtain fixed point theorems for that class.

The condition $(*)$ is due to Kim [10].

In our forthcoming work [23], we gave a class of maps satisfying $(*)$ as follows:

Lemma 1.4. *Let X be a convex subset of a locally convex t.v.s. Then any closed compact multimap $T : X \multimap X$ satisfies condition $(*)$.*

It is clear that the converse of Lemma 1.4 does not hold.

2. Maps having the a.f.p.p.

In this section, we are mainly concerned with mapping classes having the a.f.p.p.

A nonempty subset Y of E is said to be *almost convex* [8] if for any neighborhood V of the origin 0 of E and for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of Y, there exists a finite subset $\{z_1, z_2, \cdots, z_n\}$ of Y, such that $z_i - y_i \in V$ for each $i = 1, \cdots, n$ and $\mathrm{co}\{z_1, z_2, \cdots, z_n\} \subset Y$.

From the KKM principle, we obtained the following [24]:

Theorem 2.1. *Let X be a subset of a t.v.s. E and Y an almost convex subset of X. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(y)$ is convex for all $y \in Y$. Suppose that*

(Z_1) *for each neighborhood U of 0 in E, there exists a neighborhood V of 0 in E such that*

$$\mathrm{co}(V \cap (T(Y) - Y)) \subset U.$$

If there is a precompact subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$ and $Y \cap K$ is dense in K, then T has the a.f.p.p.

From Theorems 1.1 and 2.1, we deduced a number of fixed point theorems for compact upper semicontinuous maps; see [24]. The following is an example:

Theorem 2.2. *Let X be a convex subset of a t.v.s. E. Then any compact upper semicontinuous multimap $T : X \multimap X$ with nonempty closed convex values has a fixed point in X whenever the following holds:*

(Z_2) *for each neighborhood U of 0 in E, there exists a neighborhood V of 0 in E such that*

$$\text{co}(V \cap (T(X) - T(X))) \subset U.$$

A map T satisfying condition (Z_2) is usually said to be of *the Zima type* and its study was initiated by Hadžić; see [7].

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0, 1)$ of equivalence classes of measurable functions on $[0, 1]$, and others. Moreover, any locally convex subset of an F-normable t.v.s. and any compact convex locally convex subset of a t.v.s. is admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space l^2 is known. For details, see Hadžić [7], Weber [25], and references therein.

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is any convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

We define the *"better" admissible class* $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that for any polytope P in X and any continuous map $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

Examples of subfamilies of $\mathfrak{B}$ are as follows: the class $\mathbb{C}$ of single-valued continuous maps, the class $\mathbb{K}$ of the Kakutani maps (u.s.c. with compact convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (u.s.c. with compact R_δ values), the acyclic maps $\mathbb{V}$ (u.s.c. with compact acyclic values), the Powers maps $\mathbb{V}_c$ (the compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values consisting of one or more m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (in uniform spaces) or approximable maps $\mathbb{A}^\kappa$ due to Ben-El-Mechaiekh and Idzik, admissible maps in the sense of Górniewicz, permissible maps of Dzedzej, the admissible map $\mathfrak{A}_c^\kappa$ of Park, and others. For details, see [16, 17].

The following fixed point theorem was obtained recently by the author [16, 17]:

Theorem 2.3. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $T \in \mathfrak{B}(X, X)$ has the a.f.p.p. Further, if T is closed, then T has a fixed point.*

Let $\langle X \rangle$ denote the set of all nonempty finite subsets of a set X.

Let X be a convex subset of a linear space and Y a topological space. In 1996, Chang and Yen [5] defined

$T \in \mathrm{KKM}(X, Y) \iff T : X \multimap Y$ is a map such that the family $\{S(x) : x \in X\}$ has the finite intersection property whenever $S : X \multimap Y$ has closed values and $T(\mathrm{co}\, N) \subset S(N)$ for each $N \in \langle X \rangle$.

Moreover, Chang and Yen [6] introduced the class of S-KKM maps and gave a characterization of such maps and an s-KKM theorem for $s : X \to Y$. This was extended to S-KKM maps by Lin and Chang [11] with additional results. This is followed by Chang, Huang, Jeng, and Kuo [4].

Let X be a nonempty set, Y a nonempty convex subset of a linear space and Z a topological space. If $S : X \multimap Y$, $T : Y \multimap Z$ and $F : X \multimap Z$ are three multimaps satisfying

$$T(\mathrm{co}\, S(A)) \subset F(A)$$

for any $A \in \langle X \rangle$, then F is called a *generalized S-KKM map* with respect to T. If the multimap $T : Y \multimap Z$ satisfies that for any generalized S-KKM map F with respect to T the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the *S-KKM property*. The class S-KKM(X, Y, Z) is defined to be the set $\{T : Y \multimap Z \,|\, T$ has the S-KKM property$\}$; see [6, 11].

As shown in [4], if $X = Y$ and S is the identity map 1_X, then S-KKM(X, X, Z) reduces to the class KKM(X, Z) introduced by Chang and Yen [5], and moreover, KKM(Y, Z) is contained in S-KKM(X, Y, Z) for any $S : X \multimap Y$ and generally this inclusion is proper.

All of the authors of [3-6, 11, 12] claim that their multimap classes properly contain the class $\mathfrak{A}_c^\kappa$ under some restriction by giving a few examples which are trivial but neither substantial nor practical.

If S is a single-valued map $s : X \to Y$, then the class s-KKM(X, Y, Z) were treated in [9]. We know that certain s-KKM classes are contained in the class $\mathfrak{B}$; see [23].

Lemma 2.4. *Let X be a convex subset of a t.v.s., I a set, $s : I \to X$ a surjection, and $T \in s$-KKM(I, X, X). If T is closed and compact, then $T \in \mathfrak{B}(X, X)$.*

From Lemma 2.4 and Theorem 2.3, we have the following [23]:

Theorem 2.5. *Let E be a t.v.s. and X an admissible (in the sense of Klee) convex subset of E, I a set, $s : I \to X$ a surjection, and $T \in s\text{-KKM}(I, X, X)$. If T is closed and compact, then T has a fixed point.*

Note that if I is a nonempty subset of X, then Theorem 2.5 reduces to Chang, Huang, and Jeng [3, Theorem 3.1].

It should be noticed that the main fixed point theorems in [3, 5, 12] and others are disguised forms of our Theorem 2.3. Most of other results in those papers are also formally generalized (but not practical) or disguised forms of earlier works of the author on the classes $\mathfrak{A}_c^\kappa$ or $\mathfrak{B}$ of multimaps.

3. Maps having the c.a.f.p.p.

In this section, we are mainly concerned with mapping classes having the c.a.f.p.p.

From Theorem 2.1, we deduced the following in [24]:

Theorem 3.1. *Let X be a subset of a t.v.s. E and Y an almost convex subset of X. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(y)$ is convex for all $y \in Y$. If there is a precompact subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$ and $Y \cap K$ is dense in K, then T has the c.a.f.p.p.*

From Theorem 3.1, we have some particular cases as follows and, from each of them, we can deduce fixed point theorems for closed compact maps defined on convex subsets of a locally convex t.v.s.

In case $X = Y$, Theorem 3.1 reduces to the following:

Corollary 3.2. *Let X be a convex subset of a t.v.s. E. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(x)$ is convex for each $x \in X$. If there is a precompact subset K of X such that $T(x) \cap K \neq \emptyset$ for each $x \in X$, then T has the c.a.f.p.p.*

From Theorem 3.1, we have

Corollary 3.3. *Let X be a subset of a t.v.s. E and Y an almost convex dense subset of X. Let $T : X \multimap E$ be a multimap such that* (1) *$T^{-}(z)$ is open for each $z \in E$; and* (2) *$T(y)$ is convex for each $y \in Y$. If there is a precompact subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$ and $Y \cap K$ is dense in K, then T has the c.a.f.p.p.*

In case $X = Y$, Corollary 3.3 reduces to the following:

Corollary 3.4. *Let X be a convex subset of a t.v.s. E, and $T : X \multimap X$ be a multimap such that* (1) *$T(x)$ is convex for each $x \in X$;* (2) *$T^{-}(y)$ is open for each $y \in X$; and* (3) *$T(X)$ is contained in a compact subset K of X. Then T has the c.a.f.p.p.*

Ben-El-Mechaiekh [1, 2] showed that, if E is further assumed to be locally convex in Corollary 3.4, then T has a fixed point; and conjectured that, under the hypotheses of Corollary 3.4, T would have a fixed point. This conjecture is not resolved yet; for partial solutions, see [19]. However, Corollary 3.4 is a new partial solution.

We know that certain s-KKM classes have the c.a.f.p.p.; see [23]:

Theorem 3.5. *Let I be a nonempty set, X a convex subset of a t.v.s.* (*not necessarily Hausdorff*), *$s : I \to X$, and $T \in s$-$KKM(I, X, X)$ a compact multimap such that $T(X) \cap s(I)$ is dense in $T(X)$. Then $T : X \multimap X$ has the c.a.f.p.p.*

Combining Theorems 1.2 and 3.5, we have the following [23]:

Theorem 3.6. *Let X be a convex subset of a locally convex t.v.s., I a nonempty set, $s : I \to X$, and $T \in s$-$KKM(I, X, X)$ a closed compact map such that $T(X) \cap s(I)$ is dense in $T(X)$. Then T has a fixed point.*

Note that if $X = I$ and $\overline{T(X)} \subset s(I)$, then Theorem 3.6 reduces to Chang et al. [3, Theorem 3.2] and further if $s = 1_X$, then to Chang and Yen [5, Theorem 2], and if X itself is compact, then to Huang and Jeng [9, Corollary 2.4].

Combining Theorems 1.3 and 3.5, we have the following:

Theorem 3.7. *Let X be a compact convex subset of a t.v.s., I a nonempty set, $s : I \to X$, and $T \in s$-$KKM(I, X, X)$ a closed-valued map such that $T(X) \cap s(I)$ is dense in $T(X)$. If T satisfies condition* $(*)$, *then T has a fixed point.*

Note that if $X = I$ and $\overline{T(X)} \subset s(I)$, then Theorem 3.7 reduces to Huang and Jeng [9, Theorem 2.2], and if $X = I$ and $s = 1_X$, then Theorem 3.7 originates from Park [18, Theorem 1]. If $T : X \multimap X$ is upper semicontinuous with convex values, then $T \in$ KKM$(X, X) \subset s$-KKM(X, X, X); see Huang and Jeng [9].

E	$f : K \longrightarrow K$		$F : K \multimap K$	
I	Brouwer	1912	Kakutani	1941
II	Schauder	1927, 1930	Bohnenblust and Karlin	1950
III	Tychonoff Hukuhara	1935 1950	Fan Glicksberg Himmelberg	1952 1952 1972
IV	Fan	1964	Granas and Liu Park	1986 1988
V	Zima Rzepecki Hadžić	1977 1979 1982	Hadžić	1981, 1982, 1987

Finally, the above diagram shows some historically well-known fixed point theorems, where K denotes nonempty compact convex subset of a t.v.s. E, f continuous selfmaps, and F Kakutani maps (that is, upper semicontinuous maps having closed convex values).

In the diagram, E in the class I denotes Euclidean spaces, II normed vector spaces, III locally convex t.v.s., and IV topological vector spaces having sufficiently many linear functionals. Moreover, in the class V, $f(K)$ and $F(K)$ are of the Zima type. For the literature, see [13-15, 20] and other references in the end of this paper. Note that, in the diagram, theorems due to Hukuhara, Rzepecki, Himmelberg, and Hadžić are stated for compact maps without assuming compactness of domains.

It should be emphasized that each theorem in the diagram follows from one or more results in this paper, and it would be interesting to check this matter.

References

[1] H. Ben-El-Mechaiekh, *The coincidence problem for compositions of set-valued maps*, Bull. Austral. Math. Soc. **41** (1990), 421–434.

[2] H. Ben-El-Mechaiekh, *Fixed points for compact set-valued maps*, Q & A in General Topology **10** (1992), 153–156.

[3] T.-H. Chang, Y.-Y. Huang, and J.-C. Jeng, *Fixed point theorems for multifunctions in S-KKM class*, Nonlinear Anal., to appear.

[4] T.-H. Chang, Y.-Y. Huang, J.-C. Jeng, and K.-W. Kuo, *On S-KKM property and fixed point theorems*, J. Math. Anal. Appl. **229** (1999), 212–227.

[5] T.-H. Chang and C.-L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl. **203** (1996), 224–235.

[6] T.-H. Chang and C.-L. Yen, *Generalized KKM theorem and its applications*, Banyan Math. J. **2** (1996), 21–28.

[7] O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, Univ. of Novi Sad, Novi Sad, 1984, 337pp.

[8] C.J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.

[9] Y.-Y. Huang and J.-C. Jeng, *Fixed point theorems of the Park type in S-KKM class*, Nonlinear Anal. Forum, **5** (2000), 51–59.

[10] W.K. Kim, *A fixed point theorem in a Hausdorff topological vector space*, Comment. Math. Univ. Carolinae **36** (1995), 33–38.

[11] L.-J. Lin and T.-H. Chang, *S-KKM theorems, saddle points and minimax inequalities*, Nonlinear Anal. **34** (1998), 73–86.

[12] L.-J. Lin and Z.-T. Yu, *Fixed point theorems of KKM-type maps*, Nonlinear Anal. **38** (1999), 265–275.

[13] Sehie Park, *Fixed point theorems on compact convex sets in topological vector spaces*, Contemp. Math. Amer. Math. Soc. **72** (1988), 181–191.

[14] Sehie Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, ed.), World Sci., River Edge, NJ, 1992, pp.248–277.

[15] Sehie Park, *Eighty years of the Brouwer fixed point theorem*, Antipodal Points and Fixed Points (by J. Jaworowski, W.A. Kirk, and S. Park), Lect. Notes Ser. 28, RIM-GARC, Seoul Nat. Univ., 1995, pp.55–97.

[16] Sehie Park, *Fixed points of the better admissible multimaps*, Math. Sci. Res. Hot-Line **1**(9) (1997), 1–6.

[17] Sehie Park, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829. *Corrections*, ibid. **36** (1999), 829–832.

[18] Sehie Park, *Fixed point theorems for new classes of multimaps*, Acta Math. Hungar. **81** (1998), 155–161.

[19] Sehie Park, *Remarks on fixed point problems of Ben-El-Mechaiekh*, Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka, eds.), World Sci., Singapore, 1999, pp.79–86.

[20] Sehie Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. **27** (1999), 193–232.

[21] Sehie Park, *The Knaster–Kuratowski–Mazurkiewicz theorem and almost fixed points*, Top. Methods in Nonlinear Anal., to appear.

[22] Sehie Park, *The KKM theorem implies many fixed point theorems*,

[23] Sehie Park, *Remarks on fixed point theorems for new classes of multimaps*,

[24] S. Park and A. Idzik, *Almost fixed point theorems of the Zima type*,

[25] H. Weber, *Compact convex sets in non-locally convex linear spaces*, Note di Mathematica **12** (1992), 271–289.

Department of Mathematics
Seoul National University
Seoul 151–742, Korea