A UNIFIED APPROACH TO GENERALIZED KKM MAPS IN GENERALIZED CONVEX SPACES

Dedicated to Professor Ky Fan on his 80th birthday.

Sehie Park and Wanbok Lee

Abstract. In the KKM theory, many authors adopted the concept of generalized KKM maps and applied to extend or refine well-known results. In this paper, we give a unified account for such maps in generalized convex spaces. Our results include the KKM type theorems and characterizations of generalized KKM maps. We also deduce an equilibrium theorem implying minimax inequalities, variational inequalities, and so on.

1. Introduction

In 1929, Knaster, Kuratowski and Mazurkiewicz (simply, KKM) first considered the closed-valued multimap $F : D \multimap \Delta_n$, where D is the set of vertices of the standard n-simplex Δ_n, satisfying

$$\operatorname{co} A \subset F(A) \quad \text{for each } A \subset D.$$

This kind of multimaps are called the KKM map and has been extended to topological vector spaces by Fan [4], to convex spaces by Lassonde [12], to topological spaces having certain families of contractible subsets (or C-spaces or H-spaces) by Horvath [5,6], and to generalized convex spaces (simply, G-convex spaces) by Park [20-22]. In 1987, Kim [8] and Shih and Tan [28] considered the open-valued KKM maps and more refined study was given by Lassonde [13] and Park [17]. Moreover, in 1992, Tian [30] initiated the study of the so-called transfer closed-valued KKM maps and a number of authors have followed; see Park [21].

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On the other hand, Kassay and Kolumbán [9] and Chang and Zhang [2] initiated the study of the so-called generalized KKM maps and it has been followed by Chang and Ma [1], Yuan [31-33], Tan [29], Lin and Chang [15], Cheng [3], Lee, Cho, and Yuan [14], Kirk, Sims, and Yuan [10] for various classes of generalized convex spaces. All of those authors applied their results on KKM type theorems and others to extend or refine well-known results in the KKM theory [16,19]; for example, variational or quasi variational inequalities, fixed point theorems, the Ky Fan type minimax inequalities, the von Neumann type minimax or saddle point theorems, and others.

In the present paper, we are going to give a unified account for generalized KKM maps in generalized convex spaces. Our arguments are based on recent works of the first author [20,21], and much more simple and general than known works.

In Section 3, we introduce the KKM type theorems for convex spaces. Section 4 deals with the definition and characterizations of generalized KKM maps for G-convex spaces, and new KKM type theorems are deduced for such maps. In Section 5, we show that generalized KKM maps are closely related to certain general convexity of corresponding extended real-valued functions, and using this fact, deduce an equilibrium theorem implying minimax inequalities, variational inequalities, and so on.

2. The KKM principle

The following is well-known; see [8,11,12,17,19]:

The KKM Principle. *Let D be the set of vertices of Δ_n and $F : D \multimap \Delta_n$ be a KKM multimap* (*that is,* $\operatorname{co} A \subset F(A)$ *for each* $A \subset D$) *with closed* [*resp. open*] *values. Then* $\bigcap_{z\in D} F(z) \neq \emptyset$.

Here Δ_n denotes the standard n-simplex with vertices $e_0, e_1, \cdots, e_n$.

We show that the closed and the open versions are equivalent:

The open version follows from the closed version. In fact, by Shih [27, Theorem 1], for a KKM map $F : D \multimap \Delta_n$ with open values, there exists a KKM map $G : D \multimap \Delta_n$ with closed values such that $G(z) \subset F(z)$ for each $z \in D$.

Conversely, for a KKM map $F : D \multimap \Delta_n$ with closed values and for any $\varepsilon > 0$, consider a map $\varepsilon F : D \multimap \Delta_n$, where $(\varepsilon F)(z)$ is the open ε-neighborhood of $F(z)$ with respect to the Euclidean metric on Δ_n. Then εF is a KKM map with open values. Therefore, there exists an $x_\varepsilon \in \bigcap_{z\in D}(\varepsilon F)(z) \neq \emptyset$. We may assume that the net $\{x_\varepsilon\}$ converges to a limit x_0. Note that $x_0 \in \bigcap_{z\in D} F(z)$. This completes our proof.

For the history of generalizations and applications of the KKM principle, see Park [16,17,19].

3. The KKM theorem for G-convex spaces

It is known that the KKM theorem holds for topological spaces with certain abstract convexity without any linear structure. The following concept is due to the first author:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $N = \{z_0, z_1, \cdots, z_n\} \subset D$, there exists a subset $\Gamma(N) = \Gamma_N$ of X and a continuous function $\phi_N : \Delta_n \to \Gamma(N)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_N(\Delta_J) \subset \Gamma(\{z_j : j \in J\})$, where $\Delta_n = \text{co}\{e_0, e_1, \cdots, e_n\}$ is the standard n-simplex and $\Delta_J = \text{co}\{e_j : j \in J\}$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of D.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$. For a G-convex space $(X \supset D; \Gamma)$,

(1) a subset K of X is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset K$ implies $\Gamma_N \subset K$; and

(2) the Γ-*convex hull* of a subset Y of X, denoted by Γ-co(Y), is defined by

$$\Gamma\text{-co}(Y) := \bigcap\{Z \subset X : Z \text{ is a } \Gamma\text{-convex subset containing } Y\}.$$

For details on G-convex spaces and examples, see [18-26], where basic theory was extensively developed.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

Now, we deduce a KKM theorem for G-convex spaces; see [20,21]:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap such that*

(1.1) *F has closed [resp. open] values; and*

(1.2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property (More precisely, for each $N \in \langle D \rangle$, we have $\Gamma_N \cap \bigcap_{z \in N} F(z) \neq \emptyset$).

Further, if

(1.3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$.

Proof. Let $N := \{a_0, a_1, \dots, a_n\} \in \langle D \rangle$. Then there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that, for any $0 \le i_0 < i_1 < \cdots < i_k \le n$, we have

$$\phi_N(\mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\}) \cap \phi_N(\Delta_n).$$

Since F is a KKM map, it follows that

$$\begin{aligned}\mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\} &\subset \phi_N^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\}) \cap \phi_N(\Delta_n)) \\ &\subset \bigcup_{j=0}^{k} \phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n)).\end{aligned}$$

Since $F(a_{i_j}) \cap \phi_N(\Delta_n)$ is closed [resp. open] in the compact subset $\phi_N(\Delta_n)$ of Γ_N, $\phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n))$ is closed [resp. open] in Δ_n. Note that $e_i \multimap \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n))$ is a KKM map on $\{e_0, e_1, \cdots, e_n\}$. Hence, by the KKM principle, we have

$$\bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i) \cap \Gamma_N) \supset \bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n)) \neq \emptyset.$$

This readily implies $\Gamma_N \cap \bigcap_{z \in N} F(z) \neq \emptyset$. The second conclusion is clear.

For a multimap $F : D \multimap X$, we define a multimap $\overline{F} : D \multimap X$ by $\overline{F}(z) := \overline{F(z)}$ for all $z \in D$, where $\overline{}$ denotes the closure operation.

From Theorem 1, we have the following equivalent form:

Theorem 1′. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*

(1.1)′ $\bigcap_{z \in D} F(z) = \bigcap_{z \in D} \overline{F(z)}$ *[that is, F is transfer closed-valued];*
(1.2)′ $\overline{F}$ *is a KKM map; and*
(1.3)′ $\bigcap_{z \in M} \overline{F(z)}$ *is compact for some $M \in \langle D \rangle$.*

Then we have $\bigcap_{z \in D} F(z) \neq \emptyset$.

Proof. Since $\overline{F} : D \multimap X$ is closed-valued, by Theorem 1, we have $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$. Therefore, condition (1.1)′ ensures the conclusion.

Remark. This kind of KKM theorems originate from Tian [30, Theorem 2]. For some variations of Theorem 1′, see the first author's recent work [21].

4. Generalized KKM maps on G-convex spaces

Motivated by recent works on generalized KKM maps, we introduce the following definition:

Let $(X, D; \Gamma)$ be a G-convex space and I a nonempty set. A map $F : I \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N \rangle$.

Examples. (1) A generalized KKM map $F : I \multimap X$ reduces to a KKM map if $I = D$ and σ is chosen to be the identity function 1_N for each $N \in \langle D \rangle$.

(2) Kassay and Kolumbán [9] first considered the concept of generalized KKM maps. Let X and Y be convex subsets of topological vector spaces E and F, resp. A map $G : X \multimap F$ is called a generalized KKM map by Chang and Zhang [2], if for any finite set $\{x_1, \cdots, x_n\} \subset X$, there exists a finite set $\{y_1, \cdots, y_n\} \subset F$ such that any finite subset $\{y_{i_1}, \cdots, y_{i_k}\} \subset \{y_1, \cdots, y_n\}$, $1 \leq k \leq n$, we have $\mathrm{co}\{y_{i_1}, \cdots, y_{i_k}\} \subset \bigcup_{j=1}^{k} G(x_{i_j})$. Note that any KKM map $G : X \multimap E$ is a generalized KKM map, and a counterexample ensuring the converse does not hold was given in [2].

(3) Chang and Ma [1] and Kim [7] extended the preceding definition to an H-space $(X; \Gamma)$ and $I \subset X$.

(4) Yuan [31] removed the restriction $I \subset X$ in the preceding definition.

(5) Tan [29] gave a particular form of generalized KKM maps: Let $(X; \Gamma)$ be a G-convex space and I a nonempty set. A map $F : I \multimap X$ is called a generalized G-KKM map if for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to X$ such that $M \in \langle N \rangle$ implies Γ-$\mathrm{co}(\sigma(M)) \subset F(M)$. Note that all of his results in [29] were given under some superfluous restrictions.

(6) Lin and Chang [15] defined as follows: Let D be a nonempty set, $(X; \Gamma)$ a G-convex space, and $S, T : D \multimap X$. Then T is called a generalized S-KKM map if for each $N \in \langle D \rangle$, Γ-$\mathrm{co}(S(N)) \subset T(N)$. It is observed that this kind of maps can be regarded as a generalized KKM map; see Park [17]. Note that Cheng [3] considered a particular type of generalized S-KKM map.

(7) Kirk, Sims, and Yuan [10] defined generalized KKM maps for a hyperconvex metric space $X = D$.

From Theorem 1, we have the following result related to the finite intersection property for generalized KKM maps:

Theorem 2. *Let $(X, D; \Gamma)$ be a G-convex space, I a nonempty set, and $F : I \multimap X$ a map with closed [resp. open] values.*

(i) *If F is a generalized KKM map, then the family of its values has the finite intersection property (More precisely, for each $N \in \langle I \rangle$, there exists an $N' \in \langle D \rangle$ such that $\Gamma_{N'} \cap \bigcap_{z \in N} F(z) \neq \emptyset$).*

(ii) *The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$.*

Proof. (i) For each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $M \in \langle N \rangle$ implies $\Gamma_{\sigma(M)} \subset F(M)$. Let $\sigma(N)$ have $n + 1$ elements. Then there exists a continuous function $\phi_N : \Delta_n \to \Gamma_{\sigma(N)}$ such that $\phi_N(\Delta_M) \subset \Gamma_{\sigma(M)}$ for each $M \in \langle N \rangle$, where Δ_M is the face of Δ_n corresponding to $\sigma(M) \subset \sigma(N)$. Since $\Gamma_{\sigma(M)} \subset F(M) \cap \Gamma_{\sigma(N)}$, we have

$$\Delta_M \subset \phi_N^{-1}(\Gamma_{\sigma(M)}) \subset \bigcup \{\phi_N^{-1}(F(z) \cap \Gamma_{\sigma(N)}) : z \in M\}$$

for each $M \in \langle N \rangle$. Note that $F(z) \cap \Gamma_{\sigma(N)}$ is closed [resp. open] in $\Gamma_{\sigma(N)}$ and hence $\phi_N^{-1}(F(z) \cap \Gamma_{\sigma(N)})$ is closed [resp. open] in Δ_n. Moreover, $z \mapsto \phi_N^{-1}(F(z) \cap \Gamma_{\sigma(N)})$ defines a KKM map $F' : N \multimap \Delta_n$ on the G-convex space (Δ_n, N, Γ'), where $\Gamma'_M := \Delta_M$ for each $M \in \langle N \rangle$. Hence, by Theorem 1, we have

$$\bigcap_{z \in N} F'(z) = \bigcap_{z \in N} \phi_N^{-1}(F(z) \cap \Gamma_{\sigma(N)}) \neq \emptyset.$$

This readily implies

$$\Gamma_{\sigma(N)} \cap \bigcap_{z \in N} F(z) \neq \emptyset.$$

Putting $N' := \sigma(N) \in \langle D \rangle$, we have the conclusion.

(ii) Suppose that $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$. For any $N \in \langle I \rangle$, by assumption, we have an $x^* \in \bigcap_{z \in N} F(z) \neq \emptyset$. Define a function $\sigma : N \to D = X$ by $\sigma(z) = x^*$ for all $z \in N$. Then for any nonempty subset M of N, we have

$$\Gamma_{\sigma(M)} = \Gamma_{\{x^*\}} = \{x^*\} \subset \bigcap_{z \in N} F(z) \subset F(M).$$

Therefore, F is a generalized KKM map.

Examples. (1) For an H-space $(X; \Gamma)$ and $I \subset X$, Theorem 2(i) was due to Chang and Ma [1, Theorems 1 and 4].

(2) For a G-convex space $(X; \Gamma)$, a particular form of Theorem 2 was obtained by Tan [29, Theorem 2.2].

The following is a simple consequence of Theorem 2:

Theorem 3. *Let $(X;\Gamma)$ be a G-convex space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, I a nonempty set, and $F : I \multimap X$ a map with closed [resp. open] values. Then F is a generalized KKM map if and only if $\{F(z)\}_{z\in I}$ has the finite intersection property (more precisely, for each $N \in \langle I \rangle$, there exists an $N' \in \langle X \rangle$ such that $\Gamma_{N'} \cap \bigcap_{z\in N} F(z) \neq \emptyset$).*

Examples. (1) Chang and Zhang [2, Theorem 3.1] showed a closed version of Theorem 3 for the case when I is a convex subset of a Hausdorff topological vector space with the finite topology.

(2) For an H-space $(X;\Gamma)$ and $I \subset X$, the necessity of Theorem 3 was obtained by Chang and Ma [1, Theorem 1].

(3) A closed version of Theorem 3 for convex spaces was given in Park [17, Theorem 6].

(4) If X is a Hausdorff topological vector space with the finite topology, Theorem 3 reduces to Lee, Cho, and Yuan [14, Theorems 2.1, 2.2, 2.4 and Corollaries 2.5 and 2.6]. Moreover, they showed that the open version [14, Theorem 2.1] is equivalent to the closed version [14, Theorem 2.2]. However, this fact is a simple consequence of the corresponding equivalences in the KKM principle and Theorem 1.

(5) A hyperconvex metric space is a particular form of a C-space, and hence, is a G-convex space. If X is a hyperconvex metric space with finitely generated topology, Theorem 3 reduces to Kirk, Sims, and Yuan [10, Theorem 2.1] and Yuan [32, Theorems 2.2 and 2.4], who assumed superfluous restrictions. Their proofs are unnecessarily lengthy and complicated. The obvious equivalency of the closed and the open versions was also shown in [32, Theorem 2.3].

Remark. Theorems 2 and 3 hold for a transfer closed-valued map $F : I \multimap X$, that is, $\bigcap_{z\in I} F(z) = \bigcap_{z\in I} \overline{F(z)}$. In fact, in the proof of Theorem 2, we can use Theorem 1′ instead of Theorem 1.

From Theorem 2, we obtain the following characterization of generalized KKM maps:

Theorem 4. *Let $(X, D;\Gamma)$ be a G-convex space, I a nonempty set, and $F : I \multimap X$ a map with closed values such that $\bigcap_{z\in M} F(z)$ is compact for some $M \in \langle I \rangle$.*

(i) *If F is a generalized KKM map, then $\bigcap_{z\in I} F(z) \neq \emptyset$.*

(ii) *If $X = D$, $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, and $\bigcap_{z\in I} F(z) \neq \emptyset$, then F is a generalized KKM map.*

Example. Tan [29, Theorem 2.3] obtained a version of Theorem 4 for $X = D$ under some superfluous restriction.

In view of Theorem $1'$, we have a slightly general form of Theorem 4 as follows:

Theorem $4'$. *Let $(X, D; \Gamma)$ be a G-convex space, I a nonempty set, and $F : I \multimap X$ a transfer closed-valued map such that $\bigcap_{z\in M} \overline{F(z)}$ is compact for some $M \in \langle I \rangle$.*

(i) *If $\overline{F}$ is a generalized KKM map, then $\bigcap_{z\in I} F(z) \neq \emptyset$.*

(ii) *If $X = D$, $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, and $\bigcap_{z\in I} F(z) \neq \emptyset$, then F is a generalized KKM map.*

The following is a simple consequence of Theorem 4:

Theorem 5. *Let $(X; \Gamma)$ be a G-convex space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, I a nonempty set, and $F : I \multimap X$ a map with closed values such that $\bigcap_{z\in M} F(z)$ is compact for some $M \in \langle I \rangle$. Then F is a generalized KKM map if and only if $\bigcap_{z\in I} F(z) \neq \emptyset$.*

Examples. (1) Chang and Zhang [2, Theorem 3.2] obtained Theorem 5 for the case when I is a convex subset of a Hausdorff topological vector space.

(2) A convex space version of Theorem 1 was given by Park [17, Corollary 1]. Moreover, a more general compactness condition was shown to work in Theorem 5; see [17, Theorem 7].

(3) If $X = D$ is a hyperconvex metric space, Theorem 4 reduces to Kirk, Sims, and Yuan [10, Theorem 2.2].

In view of Theorem $1'$, we also have a slightly general form of Theorem 5 as follows:

Theorem $5'$. *Let $(X; \Gamma)$ be a G-convex space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, I a nonempty set, and $F : I \multimap X$ a transfer closed-valued map such that $\bigcap_{z\in M} \overline{F(z)}$ is compact for some $M \in \langle I \rangle$. Then $\overline{F}$ is a generalized KKM map if and only if $\bigcap_{z\in I} F(z)$ is nonempty compact.*

Examples. (1) Theorem $5'$ for H-spaces was first due to Zhou; see Yuan [31].

(2) For a hyperconvex metric space X, Theorem $5'$ was due to Kirk, Sims, and Yuan [10, Theorem 2.5], which is a simple consequence of the preceding one in (1).

5. Analytic formulations of generalized KKM maps

It is well-known that the KKM theory has many applications on equilibrium problems. Some applicability of our results are based on the fact that generalized KKM maps are closely related to certain convexity (or concavity) of extended real-valued functions.

Let I be a nonempty set, $(X, D; \Gamma)$ be a G-convex space, and $f : I \times X \to \overline{\mathbb{R}}$, $g : X \times I \to \overline{\mathbb{R}}$ functions. Let $\gamma \in \mathbb{R}$. Then we define as follows:

(i) f is *generalized γ-quasiconvex* [resp. *generalized γ-quasiconcave*] *on the first variable* z if for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\emptyset \neq M \subset N$ implies $\gamma \leq \max_{z \in M} f(z, x)$ [resp. $\gamma \geq \min_{z \in M} f(z, x)$] for all $x \in \Gamma_{\sigma(M)}$; and

(ii) g is *generalized γ-quasiconvex* [resp. *generalized γ-quasiconcave*] *in the second variable* z if the function $h : I \times X \to \overline{\mathbb{R}}$ defined by $h(z, x) = g(x, z)$ for all $(z, x) \in I \times X$ is generalized γ-quasiconvex [resp. generalized γ-quasiconcave] in the first variable z.

Note that, for example, $f(z, x)$ is generalized γ-quasiconvex on the first variable z if and only if $f(z, x) - \gamma$ is generalized 0-quasiconvex on the first variable.

If $I = D$ and $\sigma = i_N : N \hookrightarrow D$, the inclusion, for each $N \in \langle D \rangle$, then the above generalized γ-quasiconvexity [resp. quasiconcavity] is called *diagonally γ-quasiconvexity* [resp. *diagonally γ-quasiconcavity*].

Examples. (1) For a convex subset X of a topological vector space, the concept of generalized γ-quasiconvexity is first introduced by Chang and Zhang [2].

(2) For a convex subset X of a topological vector space and $I = D \subset X$, the concept of diagonally γ-quasiconcavity is first introduced by Tian [30] and more general than the corresponding one due to Zhou and Chen [34].

(3) For $X = D$, our generalized γ-quasiconvexity [resp. generalized γ-quasiconcavity] reduces to the γ-generalized G-quasiconvexity [resp. γ-generalized G-quasiconcavity] due to Tan [29, Definition 1.8].

(4) If $X = D$ is a hyperconvex metric space and I is a nonempty finite subset of X, Kirk–Sims–Yuan [10] defined the hyper γ-generalized quasiconvexity [resp. the corresponding quasiconcavity] as above in (2).

The following shows the equivalency of certain concavity [resp. convexity] of extended real-valued functions and the related generalized KKM maps:

Theorem 6. *Let I be a nonempty set, $(X, D; \Gamma)$ a G-convex space, $f : I \times X \to \overline{\mathbb{R}}$, and $\gamma \in \mathbb{R}$. Then the following are equivalent:*

(i) *The multimap $F : I \multimap X$, defined by $F(z) := \{x \in X : f(z, x) \leq \gamma\}$ [resp. $F(z) := \{x \in X : f(z, x) \geq \gamma\}$] for all $z \in I$, is a generalized KKM map.*

(ii) *f is generalized γ-quasiconcave [resp. generalized γ-quasiconvex] in the first variable z.*

Proof. (i) $\Rightarrow$ (ii) For any $N \in \langle I \rangle$, there exists a $\sigma : N \to D$ such that $M \in \langle N \rangle$ implies $\Gamma_{\sigma(M)} \subset F(M)$. Let $x \in \Gamma_{\sigma(M)}$. Then $x \in F(z)$ for some $z \in M$ and hence $f(z, x) \leq \gamma$

[resp. $f(z,x) \geq \gamma$]. Hence $\min_{z\in M} f(z,x) \leq \gamma$ [resp. $\max_{z\in M} f(z,x) \geq \gamma$] so that f is generalized γ-quasiconcave [resp. generalized γ-quasiconvex] in the first variable z.

(ii) $\Rightarrow$ (i) Since f is generalized γ-quasiconcave [resp. generalized γ-quasiconvex] in z, for any $N \in \langle I\rangle$, there exists a $\sigma : N \to D$ such that $M \in \langle N\rangle$ implies $\min_{z\in M} f(z,x) \leq \gamma$ [resp. $\max_{z\in M} f(z,x) \geq \gamma$] for all $x \in \Gamma_{\sigma(M)}$. Therefore, $x \in \Gamma_{\sigma(M)}$ implies $f(z,x) \leq \gamma$ [resp. $f(z,x) \geq \gamma$] for some $z \in M$, and hence $x \in F(z)$ for some $z \in M$. Therefore $\Gamma_{\sigma(M)} \subset \bigcup_{z\in M} F(z) = F(M)$ so that F is a generalized KKM map.

Examples. (1) For a convex subset $X = D$ of a topological vector space, Theorem 6 reduces to Chang and Zhang [2, Proposition 2.1].

(2) For $X = D$, Theorem 6 reduces to Tan [29, Proposition 2.1].

(3) Tan's result was obtained by Kirk–Sims–Yuan [10, Lemma 2.7] for a hyperconvex metric space X.

Corollary 6′. *Let $(X, D; \Gamma)$ be a G-convex space, $f : D \times X \to \overline{\mathbb{R}}$, and $\gamma \in \mathbb{R}$. Then the following are equivalent:*

(i) *The multimap $F : D \multimap X$, defined by $F(z) := \{x \in X : f(z,x) \leq \gamma\}$ [resp. $F(z) := \{x \in X : f(z,x) \geq \gamma\}$] for all $z \in I$, is a KKM map.*

(ii) *f is diagonally γ-quasiconcave [resp. diagonally γ-quasiconvex] in the first variable z.*

Example. If X is a convex subset of a topological vector space and $D \subset X$, Corollary 6′ is first noted by Tian [30, Remark 3].

From Theorem 6, we obtain the following equilibrium result implying minimax inequalities, variational inequalities, and so on:

Theorem 7. *Let I be a nonempty set, $(X, D; \Gamma)$ a G-convex space, $f : I \times X \to \overline{\mathbb{R}}$, and $\gamma \in \mathbb{R}$ such that*

(7.1) *for each $z \in I$, $\{x \in X : f(z,x) \leq \gamma\}$ is [transfer] closed (for example, $x \mapsto f(z,x)$ is lower semicontinuous);*

(7.2) *f is generalized γ-quasiconcave in the first variable z; and*

(7.3) *there exists a set $M \in \langle I\rangle$ such that $\bigcap_{z\in M} \overline{\{x \in X : f(z,x) \leq \gamma\}}$ is compact in X.*

Then there exists an $x_0 \in X$ such that $f(z,x_0) \leq \gamma$ for all $z \in I$.

Proof. Let us define a map $F : I \multimap X$ by $F(z) := \{x \in X : f(z,x) \leq \gamma\}$ for $z \in I$. Then, by (7.1), F is [transfer] closed-valued. By Theorem 6, (7.2) implies that F is a generalized

KKM map. Therefore, by Theorem 4′(i), (7.3) implies $\bigcap_{z\in I} F(z) \neq \emptyset$. Hence there exists an $x_0 \in X$ such that $x_0 \in F(z)$ or $f(z, x_0) \leq \gamma$ for all $z \in I$. This completes our proof.

Examples. (1) For closed convex subsets I and $X = D$ of Hausdorff topological vector spaces, Theorem 7 reduces to Chang and Zhang [2, Theorem 3.4].

(2) Tan [29, Theorem 3.1] obtained Theorem 7 for $X = D$ under some superfluous restrictions.

(3) Tan's result was obtained by Kirk–Sims–Yuan [10, Theorem 2.8] for a hyperconvex metric space X.

Final Remarks. (1) From a particular form [2, Theorem 3.4] of Theorem 7, Chang and Zhang obtained a general variational inequality of the Browder–Hartman–Stampacchia type, a saddle point theorem, an existence theorem of solutions for a class of quasi-variational inequalities, and a generalization of the Fan–Glicksberg fixed point theorem.

(2) From a particular form [29, Theorem 3.1] of Theorem 7, Tan deduced the Ky Fan type minimax inequalities [29, Theorems 3.2, 3.4, and Proposition 3.4], and several existence results of saddle points and minimax theorems [29, Theorems 4.1, 4.2, 4.4-4.6]. Those results can also be generalized and improved by adopting our method in this paper.

(3) Most of the results in Kirk–Sims–Yuan [10] and Yuan [32] are already known for H-spaces or G-convex spaces.

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Sehie Park
Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: `shpark@math.snu.ac.kr`

Wanbok Lee

Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: wblee@math.snu.ac.kr