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New topological versions of the Fan–Browder fixed point theorem

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Abstract

We give several topological versions of the Fan–Browder fixed point theorem. In fact, we obtain a number of generalizations of the theorem for G-convex spaces and ω-connected spaces, and some variants for Φ-spaces and for complete LC-metric spaces. Several coincidence or fixed point theorems related to the theorem are given. One of our new theorems is applied to generalize the Fan intersection theorem and the Nash equilibrium theorem.

1. Introduction

In 1961, using his own generalization of the Knaster–Kuratowski–Mazurkiewicz (simply, KKM) theorem, Ky Fan [7] established an elementary but very basic "geometric" lemma for multimaps and gave several applications. In 1968, Browder [4] obtained a fixed point theorem which is the more convenient form of Fan's lemma. With this result alone, Browder carried through a complete treatment of a wide range of coincidence and fixed point theory, minimax theory, variational inequalities, monotone operators, and game theory. Since then this result is known as the Fan-Browder fixed point theorem, and there have appeared numerous generalizations and new applications. For the literature, see [18-23, 26-28].

At first the Fan-Browder theorem [4] was given for Hausdorff topological vector spaces (simply, t.v.s.), and, later, extended for convex spaces. A *convex space* is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets; see Lassonde [15]. Such convex hulls are called *polytopes*. The concept of convex spaces was extended to that of C-spaces (or H-spaces) by Horvath [10-13], and further to G-convex spaces by the author [20-25,27,28].

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The main purpose in this paper is to give generalizations or variants of the Fan–Browder theorem replacing the convexity by some adequate concepts in topology (for example, contractibility or ω-connectedness). In fact, we give several generalizations of the theorem: coincidence theorems and fixed point theorems for G-convex spaces (Theorems 3.1 and 4.1) and for ω-connected spaces (Theorems 5.1-5.3). Other variants of the theorem are added for Φ-spaces (Theorem 6.4) and for complete LC-metric spaces (Theorems 7.2 and 7.3). One of our new theorems is applied to the Fan intersection theorem (Theorem 4.6) and the Nash equilibrium theorem (Theorem 4.7) for G-convex spaces.

Our arguments are based on a KKM type theorem for G-convex spaces [23,24].

2. Preliminaries

A *multimap* or *map* $F : X \multimap Y$ is a function from a set X into the power set of a set Y. Note that $F^-(y) := \{x \in X : y \in F(x)\}$ for $y \in Y$ and $F(A) := \bigcup_{x \in A} F(x)$ for $A \subset X$.

For topological spaces X and Y, an *admissible class* $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is one such that, for each T and each compact subset K of X, there exists a map $F \in \mathfrak{A}_c(K, Y)$ satisfying $F(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous with nonempty compact values; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

The notions of convex spaces and C-spaces were extended by the author as follows [23,25,27,28]:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $N = \{z_0, z_1, \cdots, z_n\} \subset D$, there exist a subset $\Gamma(N) = \Gamma_N$ of X and a continuous function $\phi_N : \Delta_n \to \Gamma(N)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_N(\Delta_J) \subset \Gamma(\{z_j : j \in J\})$, where $\Delta_n = \mathrm{co}\{e_0, e_1, \cdots, e_n\}$ is the standard n-simplex and $\Delta_J = \mathrm{co}\{e_j : j \in J\}$.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of D.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$; and for any subset $Y \subset X$, the Γ-*convex hull* of Y is defined by as follows:

$$\Gamma\text{-co}\, Y := \bigcap \{Z \subset X : Z \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } Y\}.$$

It is easily seen that $\Gamma\text{-co}\, Y = \bigcup \{\Gamma\text{-co}\, N : N \in \langle Y \rangle\}$.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is called a *G-convex subspace* if $(Y, Y \cap D; \Gamma')$ is a G-convex space where $\Gamma'_A := \Gamma_A \cap Y$ for $A \in \langle Y \cap D \rangle$.

Examples of admissible classes of maps and G-convex spaces are given in [22,25-28].

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if

$$\Gamma_N \subset F(N) \quad \text{for each } N \in \langle D \rangle.$$

The following is a KKM theorem for G-convex spaces [23,24]:

Theorem 2.1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*
(1) *F has closed [resp. open] values; and*
(2) *F is a KKM map.*
Then $\{F(z)\}_{z \in D}$ has the finite intersection property. Further, if
(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*
then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

3. Coincidence Theorems

Motivated by the earlier works of the present author on the class $\mathfrak{A}_c^\kappa$, Chang and Yen [6] introduced the more general class of multimaps having the KKM property. This concept is further extended by Lin et al. [17] as follows:

Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. A multimap $F : X \multimap Y$ is said to have *the KKM property* if, for any map $G : D \multimap Y$ with closed values satisfying

$$F(\Gamma_A) \subset G(A) \qquad \text{for all} \quad A \in \langle D \rangle,$$

the family $\{G(z)\}_{z \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(X, Y) := \{F : X \multimap Y \mid F \text{ has the KKM property}\}.$$

Some authors use the notation $KKM(X, Y)$. Note that $1_X \in \mathfrak{K}(X, X)$ by Theorem 2.1.

We immediately have the following basic coincidence theorem:

Theorem 3.1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a topological space, $S : D \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{K}(X, Y)$. Suppose that*
(1) *S has open values;*
(2) *for each $y \in F(X)$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$\overline{F(X)} \subset S(N)$ for some $N \in \langle D \rangle$.*
Then F and T have a coincidence point $x_* \in X$; that is, $F(x_*) \cap T(x_*) \neq \emptyset$.

Proof. Define a map $G : D \multimap Y$ by $G(z) := \overline{F(X)} \backslash S(z)$ for each $z \in D$. Then G has closed values and

$$\bigcap_{z \in N} G(z) = \bigcap_{z \in N} (\overline{F(X)} \backslash S(z)) = \overline{F(X)} \backslash \bigcup_{z \in N} S(z) \subset \overline{F(X)} \backslash \overline{F(X)} = \emptyset.$$

Therefore $\{G(z)\}_{z \in D}$ does not have the finite intersection property.

Since $F \in \mathfrak{K}(X, Y)$, by definition, we have

$$F(\Gamma_A) \not\subset G(A) \qquad \text{for some} \quad A \in \langle D \rangle.$$

Hence, there exists a $y_0 \in F(\Gamma_A) \subset F(X)$ such that $y_0 \notin G(z) = \overline{F(X)} \backslash S(z)$ for all $z \in A$. Therefore, $y_0 \in S(z)$ or $z \in S^-(y_0)$ for all $z \in A$ and hence $A \in \langle S^-(y_0) \rangle$. Since $y_0 \in F(X)$, by (2), we have $\Gamma_A \subset T^-(y_0)$ and hence $y_0 \in F(\Gamma_A) \subset F(T^-(y_0))$.

Therefore, there exists an $x_* \in T^-(y_0)$ such that $y_0 \in F(x_*)$, and hence we have $y_0 \in F(x_*) \cap T(x_*)$. This completes our proof.

Remarks 1. Note that condition (3) is satisfied if the following holds:
(3)′ *F is compact and $\overline{F(X)} \subset S(D)$.*
2. Instead of (1), the following suffices:
(1)′ *$\overline{F(X)} \cap S(z)$ is open in $\overline{F(X)}$ for each $z \in D$.*
3. Theorem 3.1 generalizes [6, Theorem 4].

4. The Fan–Browder fixed point theorem

We have a general form of the Fan-Browder theorem for G-convex spaces:

Theorem 4.1. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ two maps satisfying*
(1) *for each $z \in D$, $S(z)$ is open [resp. closed];*
(2) *for each $y \in X$, $M \in \langle S^-(y)\rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$X = S(N)$ for some $N \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

Proof. Let $F : D \multimap X$ be defined by $F(z) := X\backslash S(z)$ for $z \in D$. Then each $F(z)$ is closed [resp. open] by (1). Moreover, by (3), we have

$$\bigcap_{z\in N} F(z) = X\backslash \bigcup_{z\in N} S(z) = X\backslash X = \emptyset,$$

and hence, $\{F(z)\}_{z\in D}$ does not have the finite intersection property. Therefore, by Theorem 2.1, F can not be a KKM map; and hence, there exist an $M \in \langle D\rangle$ and an $x_* \in \Gamma_M$ such that $x_* \notin F(M)$. Since $x_* \notin F(z)$ or $x_* \in S(z)$ for all $z \in M$, we have $z \in S^-(x_*)$ for all $z \in M$ and hence $M \in \langle S^-(x_*)\rangle$. Therefore, by (2), we have $\Gamma_M \subset T^-(x_*)$. Since $x_* \in \Gamma_M$, we have $x_* \in T^-(x_*)$. This completes our proof.

Remarks 1. For the case when S has open values, condition (3) is satisfied if
(3)′ *X is compact and $S(D) = X$.*
2. For the case when S has open values, Theorem 4.1 follows from Theorem 3.1 for $X = Y$ and $F = 1_X$.

From Theorem 4.1, we deduced the following generalization of the Fan–Browder fixed point theorem in [24]:

Theorem 4.2. *Let $(X \supset D; \Gamma)$ be a G-convex space, K a nonempty subset of X, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that*
(1) *for each $x \in X$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$;*
(2) *$K \subset \bigcup\{\operatorname{Int} S^-(z) : z \in N\}$ for some $N \in \langle D\rangle$; and*
(3) *there exists a G-convex subspace L_N of X containing N such that*

$$L_N\backslash K \subset \bigcup\{\operatorname{Int} S^-(z) : z \in M\} \quad \text{for some } M \in \langle L_N \cap D\rangle.$$

Then T has a fixed point in L_N.

From Theorem 4.1 or 4.2, we have the following:

Corollary 4.3. *Let $(X \supset D;\Gamma)$ be a G-convex space, and $F : X \multimap D$ a map satisfying $X = \bigcup\{\operatorname{Int} F^-(x) : x \in N\}$ for some $N \in \langle D\rangle$. Then the map $\Gamma\text{-co}\, F : X \multimap X$, defined by $(\Gamma\text{-co}\, F)(x) = \Gamma\text{-co}F(x)$ for $x \in X$, has a fixed point.*

Other particular forms of Theorem 4.2 can be found in [20]. Especially, Theorem 4.1 for $X = D$ reduces to the following generalization of the Fan-Browder theorem:

Corollary 4.4. *Let X be a convex space, and $S, T : X \multimap X$ two maps satisfying*

(1) *for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and*
(2) *$X = \bigcup\{\operatorname{Int} S^- y : y \in N\}$ for some $N \in \langle X\rangle$.*

Then T has a fixed point $x_0 \in X$.

Popular generalizations of the Fan–Browder theorem have the form of Theorem 4.1 for $X = D$ and $S = T$ as follows:

Corollary 4.5. *Let $(X;\Gamma)$ be a compact G-convex space and $T : X \multimap X$ a map such that*

(1) *$T(x)$ is nonempty Γ-convex for each $x \in X$; and*
(2) *$T^-(y)$ is open for each $y \in X$.*

Then T has a fixed point.

In the remainder of this section, we give simple applications of Corollary 4.5.

Given a Cartesian product $X = \Pi_{i=1}^n X_i$ of topological spaces, let $X^i = \Pi_{j\neq i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections. We write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$(y_i, x^i) = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

Theorem 4.6 (The Ky Fan intersection theorem). *Let $(X;\Gamma)$ be a compact G-convex space with $X = \Pi_{i=1}^n X_i$ and $A_1, A_2, \ldots, A_n$ be n subsets of X such that*

(1) *for each $x \in X$ and each $i = 1, \ldots, n$, the set $A_i(x) = \{y \in X : (y_i, x^i) \in A_i\}$ is Γ-convex and nonempty; and*
(2) *for each $y \in X$ and each $i = 1, \ldots, n$, the set $A_i(y) = \{x \in X : (y_i, x^i) \in A_i\}$ is open.*

Then $\bigcap_{i=1}^n A_i \neq \emptyset$.

Proof. Define a map $T : X \multimap X$ by $T(x) = \bigcap_{i=1}^n A_i(x)$ for $x \in X$. Then each $T(x)$ is Γ-convex being an intersection of Γ-convex sets by (1). For each $x \in X$ and each i, there exists a $y^{(i)} \in A_i(x)$ by (1), or $(y_i^{(i)}, x^i) \in A_i$. Hence, we have $(y_1^{(1)}, \ldots, y_n^{(n)}) \in \bigcap_{i=1}^n A_i(x)$. This shows $T(x) \neq \emptyset$. Moreover, $T^-(y) = \bigcap_{i=1}^n A_i(y)$ is open for each $y \in X$ by (2). Now, the conclusion follows from Corollary 4.5.

Remarks 1. If each X_i is a compact G-convex space, so is X.

2. In view of Corollary 4.4, condition (2) can be replaced by the following:

(2)′ $X = \bigcup_{y\in X} \operatorname{Int}(\bigcap_{i=1}^n A_i(y))$.

Theorem 4.7 (The Nash equilibrium theorem). *Let $(X;\Gamma)$ be a compact G-convex space with $X = \Pi_{i=1}^n X_i$ and $f_1, \dots, f_n : X \to \mathbf{R}$ continuous functions such that*

(3) *for each $x \in X$, each $i = 1, \dots, n$, and each $r \in \mathbf{R}$, the set $\{(y_i, x^i) \in X : f_i(y_i, x^i) > r\}$ is Γ-convex.*

Then there exists a point $x \in X$ such that

$$f_i(x) = \max_{y_i \in X_i} f_i(y_i, x^i) \quad \text{for } i = 1, \dots, n.$$

Proof. Let $\varepsilon > 0$ and, for each i, let

$$A_i^\varepsilon = \{x \in X : f_i(x) > \max_{y_i \in X_i} f_i(y_i, x^i) - \varepsilon\}.$$

Then the sets $A_1^\varepsilon, \dots, A_n^\varepsilon$ satisfy conditions (1) and (2) of Theorem 4.6, and hence $\bigcap_{i=1}^n A_i^\varepsilon \neq \emptyset$. Then $H_\varepsilon = \bigcap_{i=1}^n \overline{A_i^\varepsilon}$ is a nonempty compact set. Since $H_{\varepsilon_1} \subset H_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$, we have $\bigcap_{\varepsilon>0} H_\varepsilon \neq \emptyset$. Then $x \in \bigcap_{\varepsilon>0} H_\varepsilon$ satisfies the conclusion.

Remark. Theorems 4.6 and 4.7 are sharpened versions of [3, Theorems 4.15 and 4.16]. The point $x = (x_1, x_2, \cdots, x_n)$ in the conclusion of Theorem 4.7 is called a Nash point equilibrium. This concept is a natural extension of the local maxima (for the case $n = 1$, $f = f_1$) and of the saddle points (for the case $n = 2$, $f_1 = -f$, $f_2 = f$)

5. For ω-connected spaces

An *ω-connected space* X is a topological space which is n-connected for all $n \geq 0$ (or *infinitely connected*; that is, any continuous function defined on the boundary of a finite dimensional ball with values in X can be extended to a continuous function on the ball with values in X).

We give examples of ω-connected spaces as follows: (1) contractible spaces; (2) the union of two comb spaces with identifying particular points in each space; see Spanier [30]; and (3) a path-connected topological semilattice X with an element $\overline{x} \in X$ such that $x \leq \overline{x}$ for all $x \in X$ [14, Lemma 1].

A G-convex space $(X \supset D;\Gamma)$ reduces to a C-space [10-13, 26] if each Γ_A for $A \in \langle D \rangle$ is contractible (or more generally, an ω-connected subset of X) and, for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

We begin, in this section, with the following coincidence theorem:

Theorem 5.1. *Let X be an ω-connected space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that a map $G : X \multimap Y$ satisfies the following:*

(1) *for each open subset O of Y, the set $\bigcap_{y \in O} G^-(y)$ is empty or ω-connected;*

(2) *$\overline{F(X)} \subset \bigcup\{\text{Int}\, G(x) : x \in D\}$ for some $D \in \langle X \rangle$.*

Then F and G have a coincidence point.

Proof. Let us define a C-space $(X \supset D;\Gamma)$ as follows: For any $J \in \langle D \rangle$, let

$$\Gamma_J = \begin{cases} \bigcap\{G^-(y) : y \in \bigcap_{x \in J} \text{Int}\, G(x)\} & \text{if } \bigcap_{x \in J} \text{Int}\, G(x) \neq \emptyset, \\ X & \text{otherwise.} \end{cases}$$

Note that if $y \in \bigcap_{x\in J} \operatorname{Int} G(x)$, then $J \in \langle G^-(y)\rangle$. Therefore, if $O = \bigcap_{x\in J} \operatorname{Int} G(x) \neq \emptyset$, then $\Gamma_J = \bigcap_{y\in O} G^-(y)$ is an ω-connected set by (1). Moreover, it is clear that $\Gamma_J \subset \Gamma_{J'}$, whenever $J \subset J' \in \langle D\rangle$.

Now $(X \supset D;\Gamma)$ is a G-convex space. Since Y is Hausdorff, we have $\mathfrak{A}_c^\kappa(X,Y) \subset \mathfrak{K}(X,Y)$ [28]. We apply Theorem 3.1 with $S = (\operatorname{Int} G)|_D$ and $T = G$. Then conditions (1) and (3) of Theorem 3.1 are readily satisfied. We show that condition (2) of Theorem 3.1 holds. In fact, for each $y \in Y$ and $M \in \langle S^-(y)\rangle \subset \langle D \cap G^-(y)\rangle$, we have $y \in \bigcap_{x\in M} S(x) = \bigcap_{x\in M} \operatorname{Int} G(x) \neq \emptyset$. Hence $\Gamma_M = \bigcap\{G^-(z) : z \in \bigcap_{x\in M} \operatorname{Int} G(x)\} \subset G^-(y) = T^-(y)$. Therefore, by Theorem 3.1, F and G have a coincidence point.

Remarks. 1. If F is single-valued, then Y can be a mere topological space; see [28].

2. If X is a convex space, then condition (1) of Theorem 5.1 can be replaced by

(1)′ *for each $y \in Y$, $G^-(y)$ is convex.*

In this case, Theorem 5.1 reduces to a result equivalent to [19, Theorem 5], which extends many known theorems and has numerous applications in the KKM theory; see [19].

3. If ω-connected sets are replaced by contractible sets, Theorem 5.1 improves Park and Jeong [29, Theorem 2].

If F is a compact map in Theorem 5.1, we have the following:

Theorem 5.2. *Let X be an ω-connected space, Y a Hausdorff compact space, and $S \in \mathfrak{A}_c^\kappa(X,Y)$. Suppose that $T : Y \multimap X$ is a map such that*

(1) *for each open set O in Y, the set $\bigcap_{y\in O} T(y)$ is empty or ω-connected; and*

(2) $Y = \bigcup\{\operatorname{Int} T^-(x) : x \in X\}$.

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in S(x_0)$ and $x_0 \in T(y_0)$.

Proof. Let $F = S$, $Y = K$, and $T = G^-$ in Theorem 5.1. Then S and T^- have a coincidence point $x_0 \in X$ and a point $y_0 \in S(x_0) \cap T^-(x_0)$. This completes our proof.

Remarks 1. Theorem 5.2 originates from Browder [4, Theorem 7], where X is a convex subset of a t.v.s., S is a Kakutani map (that is, upper semicontinuous with nonempty closed convex values), and T has convex values.

2. Tarafdar and Yuan [33, Theorem 1] obtained Theorem 5.2 replacing ω-connected sets by contractible sets and assuming S is an upper semicontinuous map with compact contractible values. Their result is later applied in [34] to existence of fixed points, maximal elements, and equilibria for qualitative and generalized games.

3. Replacing ω-connected sets by contractible sets, Theorem 5.2 reduces to Park and Jeong [29, Theorem 3].

The following simple consequence of Theorem 5.2 generalizes the Fan-Browder theorem:

Theorem 5.3. *Let X be a compact ω-connected space and $G : X \multimap X$ a map satisfying*

(1) *for each open set O in X, the set $\bigcap_{x\in O} G(x)$ is empty or ω-connected; and*

(2) $X = \bigcup\{\operatorname{Int} G^-(x) : x \in X\}$.

Then G has a fixed point $x_0 \in X$.

Remarks. 1. If ω-connected sets are replaced by convex sets, Theorem 5.3 reduces to the Fan-Browder fixed point theorem.

2. If ω-connected sets are replaced by contractible sets, Theorem 5.3 contains results due to Horvath [9-12]; see also [29, Theorem 4].

3. Theorem 5.3 sharpens Bielawski [3, Corollary 4.10].

6. For Φ-spaces

In order to give another topological version of the Fan-Browder theorem, we generalize some concepts in [12, Section 4] as follows:

Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. Then a map $T : Y \multimap X$ is called a *Φ-map* if there is a map $S : Y \multimap D$ such that

(i) for each $y \in Y$, $M \in \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$; and
(ii) $Y = \bigcup\{\operatorname{Int} S^-(x) : x \in D\}$.

A G-convex space $(X, D; \Gamma)$ is called a *Φ-space* if X is a Hausdorff uniform space and for each entourage V there is a Φ-map $T : X \multimap X$ such that $\operatorname{Gr}(T) \subset V$. Every nonempty convex subset of a locally convex t.v.s. is a Φ-space. For other examples, see [12].

For Φ-maps, we obtained the following selection theorem in our previous work [20, Theorem 2]:

Lemma 6.1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff compact space, and $T : Y \multimap X$ a Φ-map. Then T has a continuous selection $f : Y \to X$; that is, $f(y) \in T(y)$ for all $y \in Y$. More precisely, there exist two continuous functions $p : Y \to \Delta_n$ and $\phi : \Delta_n \to X$ such that $f = \phi \circ p$ and $f(Y) \subset \Gamma_A$ for some $A \in \langle D \rangle$ having $n+1$ elements.*

Therefore, every Φ-map belongs to $\mathfrak{C}_c^\kappa(Y, X) \subset \mathfrak{A}_c^\kappa(Y, X)$ if Y is Hausdorff.

Note that Lemma 6.1 sharpens the compact case of [12, Theorem 3.2], [32, Theorem 1.2], and [31, Theorems 2.3 and 2.4].

From Theorem 3.1, we can deduce the following:

Theorem 6.2. (1) *Let $(X, D; \Gamma)$ be a G-convex space, Y a topological space, and $T : Y \multimap X$ is a Φ-map. If $g : X \to Y$ is a compact continuous map, then there exists an $x_0 \in X$ such that $x_0 \in Tg(x_0)$.*

(2) *If $(X, D; \Gamma)$ is a Φ-space, then any compact continuous map $g : X \to X$ has a fixed point.*

Proof. (1) We apply Theorem 3.1. Since T is a Φ-map, there exists a companion map $S : Y \multimap D$ satisfying (i) and (ii). Let $\hat{S} : D \multimap Y$ be a map such that $\hat{S}(x) = \operatorname{Int} S^-(x)$ for $x \in D$. Then $\hat{S}(x)$ is open in Y for each $x \in D$. Moreover, for each $y \in Y$, $M \in \langle \hat{S}^-(y) \rangle \subset \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$ by (i). Therefore, all of the requirements of Theorem 3.1 replacing (F, S, T) by $(g, \hat{S}, T^-)$ are satisfied. Hence g and T^- have a coincidence point $x_0 \in X$; that is, $g(x_0) \in T^-(x_0)$. This completes our proof of (1).

(2) For any entourage V of the uniform structure and any Φ-map $T : X \multimap X$ such that $\operatorname{Gr}(T) \subset V$, there is a point $x_V \in X$ such that $(g(x_V), x_V) \in V$ by (1). Therefore,

g has a V-fixed point for any entourage V. Since $\overline{g(X)}$ is compact, g must have a fixed point.

Remark. Theorem 6.2 sharpens [12, Theorems 4.3 and 4.4], where $(X;\Gamma)$ is a C-space and $S(x) \subset T(x)$ for all $x \in X$. Note that Theorem 6.2 also can be deduced from Lemma 6.1. However, in this case, we have to assume the Hausdorffness of Y.

For a paracompact space, we have the following selection theorem due to the author [21, Theorem 8]:

Lemma 6.3. *Let $(X \supset D;\Gamma)$ be a C-space, Y a paracompact space, and $T : Y \multimap X$ a Φ-map. Then T has a continuous selection.*

For $X = D$ and $T = S$, Lemma 6.3 reduces to Horvath [13, Theorem 3], which was applied to interesting selection problems.

From Theorem 6.2(2) and Lemma 6.3, we have the following variant of the Fan–Browder theorem:

Theorem 6.4. *Let $(X \supset D;\Gamma)$ be a paracompact C-space. If it is also a Φ-space, then any compact Φ-map $T : X \multimap X$ has a fixed point.*

Proof. By Lemma 6.3, T has a continuous selection $f : X \to X$. Since $f(X) \subset \overline{T(X)}$, f is compact. Therefore, by Theorem 6.2(2), f has a fixed point $x_0 = f(x_0) \in T(x_0)$. This completes our proof.

We give two examples of Theorem 6.4 as follows:

Corollary 6.5. *Let E be a locally convex t.v.s., X a nonempty convex subset of E, and D a dense subset of X. Then any compact Φ-map $F : X \multimap X$ has a fixed point.*

Proof. Note that $(X \supset D;\Gamma)$ with $\Gamma =$co is a C-space. Let $K = \text{co}\,\overline{F(X)}$. Then $K \subset X$ since $\overline{F(X)} \subset X$ and X is convex. Also K is σ-compact (see Fournier and Granas [8] or Lassonde [16]) and hence K is Lindelöf. Since K is regular as a subset of a t.v.s., we know that K is paracompact. Moreover, clearly $(K, D \cap K;\Gamma)$ is a C-space.

We know that $(K, D \cap K;\Gamma)$ is a Φ-space. Let V be a symmetric convex open neighborhood of 0 in E. Define $S : K \multimap D \cap K$ and $T : K \multimap K$ by

$$S(x) = (x + V) \cap D \cap K \quad \text{and} \quad T(x) = (x + V) \cap K \quad \text{for } x \in K.$$

Since D is dense in K, we have $\emptyset \neq S(x) \subset T(x)$ for each $x \in K$. Since $T(x)$ is convex, $M \in \langle S(x) \rangle$ implies $\text{co}\,M \subset T(x)$. Since D is dense in X, there is, for each $x \in K$, a $y \in D \cap K$ such that $y \in x + V$ and hence $x \in y + V$; therefore $\{y + V\}_{y \in D \cap K}$ covers K. So, for each $x \in K$, there exists a $y \in D \cap K$ such that

$$x \in (y + V) \cap K = S^-(y) = \text{Int}_K S^-(y).$$

Therefore, $K = \bigcup\{\text{Int}_K S^-(y) : y \in D \cap K\}$. Consequently, T is a Φ-map. Moreover, it is clear that for any entourage U of the uniformity of E we can find a V as above such that $\text{Gr}(T) \subset U$, and hence X is a Φ-space. Therefore, the conclusion follows from Theorem 6.4.

For $X = D$, Corollary 6.5 is well-known; see [1,22].

Lemma 6.6. *Let $(X \supset D; \Gamma)$ be a G-convex space with a metric d such that every open ball is Γ-convex and D is dense in X. Then $(X \supset D; \Gamma)$ is a Φ-space.*

Proof. Let $\varepsilon > 0$ and $S : X \multimap D$, $T : X \multimap X$ be defined by

$$S(x) = \{y \in D : d(x, y) < \varepsilon\} \quad \text{and} \quad T(x) = \{y \in X : d(x, y) < \varepsilon\}$$

for $x \in X$. Then each $T(x)$ is Γ-convex. Hence, $M \in \langle S(x) \rangle \subset \langle T(x) \rangle$ implies $\Gamma_M \subset T(x)$. Further,

$$\{x \in X : d(x, y) < \varepsilon\} = S^-(y) = \operatorname{Int} S^-(y)$$

for $y \in D$, and hence

$$X = \bigcup_{y \in D} \{x \in X : d(x, y) < \varepsilon\} = \bigcup \{\operatorname{Int} S^-(y) : y \in D\}.$$

Therefore, T is a Φ-map. Note that for any $y \in T(x)$, we have $d(x, y) < \varepsilon$. Therefore, for any entourage V of the metric uniformity, we have a Φ-map $T : X \multimap X$ such that $\operatorname{Gr}(T) \subset V$. This shows that $(X, D; \Gamma)$ is a Φ-space.

From Lemma 6.6 and Theorem 6.4, we have the following:

Corollary 6.7. *Let $(X \supset D; \Gamma)$ be a C-space with a metric d such that every open ball is Γ-convex and D is dense in X. Then any compact Φ-map $T : X \multimap X$ has a fixed point.*

Related results to Corollary 6.7 were given in Park [19], where D should be dense in X in [19, Theorem 6].

7. For LC-metric Spaces

A C-space $(X; \Gamma)$ is called an *LC-metric space* if X is equipped with a metric d such that for any $\varepsilon > 0$, the set $\{x \in X : d(x, Z) < \varepsilon\}$ is Γ-convex whenever $Z \subset X$ is Γ-convex and every open ball is Γ-convex; see [12,13].

We need the following sharp result of Ben-El-Mechaiekh and Oudadess [2, Theorem 3]:

Lemma 7.1. *Let X be paracompact, $(Y; \Gamma)$ a complete LC-metric space, $Z \subset X$ with* $\dim_X Z \leq 0$, *and $F : X \multimap Y$ lower semicontinuous with nonempty closed values such that $F(x)$ is Γ-convex for $x \notin Z$. Then F admits a continuous selection $f : X \to Y$; that is, $f(x) \in F(x)$ for $x \in X$.*

Combining Theorem 6.2(2) and Lemma 6.3, we immediately have the following:

Theorem 7.2. *Let $(X; \Gamma)$ be a complete LC-metric space, $Z \subset X$ with* $\dim_X Z \leq 0$, *and $F : X \multimap X$ a compact lower semicontinuous map with nonempty closed values such that $F(x)$ is Γ-convex for $x \notin Z$. Then F has a fixed point.*

Proof. As in Lemma 6.6, $(X; \Gamma)$ is a Φ-space. Further, by Lemma 7.1, F has a continuous selection $f : X \to X$. Note that f is compact since $\overline{f(X)} \subset \overline{F(X)}$ and F is compact. Therefore, by Theorem 6.2(2), f has a fixed point $x_0 = f(x_0) \in F(x_0)$. This completes our proof.

Remark. Particular forms of Theorem 7.2 for Banach spaces are known; for example, see [5,35].

The following is a variant of the Fan-Browder theorem:

Theorem 7.3. *Let $(X;\Gamma)$ be a complete LC-metric space, $Z \subset X$ with $\dim_X Z \leq 0$, and $F : X \multimap X$ a compact map such that*

(1) *$F(x)$ is nonempty and closed for all $x \in X$, and $F(x)$ is Γ-convex for $x \notin Z$; and*

(2) *$F^{-}(y)$ is open for all $y \in X$.*

Then F has a fixed point.

Remarks. 1. Theorem 7.3 tells that, in certain cases, the Fan–Browder theorem holds without assuming that all functional values are convex.

2. Examples of complete *LC*-metric spaces are closed convex subsets of Banach spaces, the so-called hyperconvex metric spaces, and others; see [13].

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