REMARKS ON THE HIMMELBERG-IDZIK FIXED POINT THEOREM

Sehie Park[1] and Do Hong Tan[2]

1. Seoul National University, Seoul, Korea
2. Institute of Mathematics, Hanoi, Vietnam

Abstract. In this note we give simple proofs of generalizations of the Himmelberg fixed point theorem by applying the well-known KKM principle.

1. Introduction

For a long period, the Schauder-Tychonoff fixed point theorem has been a very useful tool for the study of differential and integral equations and other fields. Motivated by von Neumann's classical works on game theory and mathematical economics, Kakutani initiated the study of fixed points of convex-valued upper semicontinuous multimaps in Euclidean spaces. The Kakutani theorem was generalized for Banach spaces by Bohnenblust and Karlin, and for locally convex topological vector spaces by Fan and by Glicksberg. Moreover, Himmelberg [1] extended and unified all of the above mentioned fixed point theorems to compact multimaps, and later, Idzik [2] further generalized those theorems for not-necessarily locally convex topological vector spaces. The Kakutani theorem and its generalizations were applied to game theory, mathematical economics, systems and control theory, coincidence theory, minimax theory, variational inequalities, convex analysis, and many equilibrium theorems. For the literature, see Park [5-7].

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

In our previous work [9], we showed that the Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem and the Schauder-Tychonoff fixed point theorem are equivalent. Our aim in this paper is to give an elementary proof of generalizations of Himmelberg fixed point theorem by applying the KKM theorem. Consequently, we show that our result is equivalent to the Brouwer fixed point theorem, which can be now proved in elementary way; for the literature, see [5], [7]. Therefore, the proofs of the above mentioned theorems (except Idzik's) should be easily accessible by every researcher or student of any fields in mathematics.

2. Preliminaries

First we recall some notions and known facts needed for our arguments. The following notion is due to Himmelberg [1] :

Definition 1. A nonempty subset Y of a topological vector space E is said to be *almost convex* if for any neighborhood V of 0 in E and for any finite set $\{y_1, y_2, \cdots, y_n\} \subset Y$, there exists a finite set $\{z_1, z_2, \cdots, z_n\} \subset Y$ such that, for each $i \in \{1, 2, \cdots, n\}$, $z_i - y_i \in V$ and $\mathrm{co}\{z_1, z_2, \cdots, z_n\} \subset Y$.

Clearly, the closure of an almost convex set is convex. Here "co" stands for the convex hull of a set.

Definition 2. Let X, Y be topological spaces. A (single- or multi-valued) map $T : X \to Y$ is called *compact* if $T(X)$ is contained in a compact subset of Y.

The following is the celebrated KKM theorem [3] :

KKM principle. *Let D be the set of vertices of a simplex S and $F : D \to 2^S$ a multimap with closed values such that*

$$\mathrm{co}\, N \subset F(N) \quad \textit{for each } N \subset D. \tag{1}$$

Then $\bigcap_{z \in D} F(z) \neq \emptyset$.

Recall that a map satisfying (1) is called a *KKM map*.

We need the following due to Shih [10, Theorem 1] :

Lemma 1. *Let D be the set of vertices of a simplex S. If $G : D \to 2^S$ is a KKM map with open values, then there is a KKM map $F : D \to 2^S$ with closed values such that $F(x) \subset G(x)$ for $x \in D$.*

From the KKM principle and Lemma 1, we get immediately

Lemma 2. *Let X be a subset of a topological vector space, D a nonempty finite subset of X such that $\operatorname{co} D \subset X$, and $F : D \to 2^X$ a KKM map with open values. Then $\bigcap_{z \in D} F(z) \neq \emptyset$.*

For the history of generalizations and applications of the open-valued version (Lemma 2) of the KKM principle, see Park *et al.* [4], [8].

3. The result

Applying Lemma 2 we shall prove the following:

Theorem 1. *Let X be a subset of a locally convex Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \to 2^X$ be a compact upper semicontinuous multimap with nonempty closed values such that $T(y)$ is convex for all $y \in Y$. Then T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.*

Proof. Let U be a convex neighborhood of the origin 0 in E. Then there exists a symmetric convex open neighborhood V of 0 such that $\overline{V} + \overline{V} \subset U$. Since $K := \overline{T(X)}$ is compact in X, there exists a finite subset $\{x_1, \cdots, x_n\} \subset K \subset X$ such that $K \subset \bigcup_{i=1}^{n} (x_i + V)$. Moreover, since Y is almost convex and dense in X, there exists a finite subset $D = \{y_1, \cdots, y_n\}$ of Y such that $y_i - x_i \in V$ for each $i = 1, \cdots, n$, and $\operatorname{co}\{y_1, \cdots, y_n\} \subset Y$. [In fact, since Y is dense in X, there is a subset $\{z_1, \cdots, z_n\}$ of Y such that $z_i - x_i \in \frac{1}{2}V$ for each i. Since Y is almost convex, there exists a subset $\{y_1, \cdots, y_n\}$ of Y such that $y_i - z_i \in \frac{1}{2}V$ for each i and $\operatorname{co}\{y_1, \cdots, y_n\} \subset Y$. Since V is convex, we have $y_i - x_i = (y_i - z_i) + (z_i - x_i) \in \frac{1}{2}V + \frac{1}{2}V \subset V$ for each i.]

For each i, let

$$F(y_i) := \{x \in X : T(x) \cap (x_i + \overline{V}) = \emptyset\}.$$

Since T is upper semicontinuous, each $F(y_i)$ is open in X. Moreover we have

$$\bigcap_{i=1}^{n} F(y_i) = \{x \in X : T(x) \cap \bigcup_{i=1}^{n}(x_i + \overline{V}) = \emptyset\} = \emptyset$$

since $T(X) \subset K \subset \bigcup_{i=1}^{n}(x_i + V)$.

Now we apply Lemma 2 to X with D defined as above. Since its conclusion does not hold, $F : D \to 2^X$ can not be a KKM map. That is, there exist a subset $\{y_{i_1}, \cdots, y_{i_k}\} \subset D$ and an $x_U \in \operatorname{co}\{y_{i_1}, \cdots, y_{i_k}\}$ such that $x_U \notin \bigcup_{j=1}^{k} F(y_{i_j})$. Hence $T(x_U) \cap (x_{i_j} + \overline{V}) \neq \emptyset$ for each j; and note that

$$x_{i_j} + \overline{V} = x_{i_j} - y_{i_j} + y_{i_j} + \overline{V} \subset y_{i_j} + V + \overline{V} \subset y_{i_j} + U. \qquad (2)$$

Let L be the subspace of E generated by D and

$$M := \{y \in L : T(x_U) \cap (y + U) \neq \emptyset\}.$$

From (2) we get $T(x_U) \cap (y_{i_j} + U) \neq \emptyset$ and hence $y_{i_j} \in M$ for all $j = 1, \cdots, k$. Since $L, T(x_U)$, and U are all convex, it is easily checked that M is convex. Therefore, $x_U \in M$ and, by definition of M, we get $T(x_U) \cap (x_U + U) \neq \emptyset$.

So, for each neighborhood U of 0, there exist $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(X)$ is relatively compact, we may assume that the net $\{y_U\}$ converges to some $x_0 \in K$. Since E is Hausdorff, the net $\{x_U\}$ also converges to x_0. Because T is upper semicontinuous with closed values, the graph of T is closed in $X \times T(X)$ and hence we have $x_0 \in T(x_0)$. This completes our proof.

In particular, for $Y = X$, we obtain

Theorem 2. *Let X be an almost convex subset of a locally convex Hausdorff topological vector space. Then any compact upper semicontinuous multimap $T : X \to 2^X$ with nonempty closed convex values has a fixed point in X.*

4. Remarks

1. For the case X itself is almost convex, Theorem 1 is a particular case of Idzik [2, Theorem 4.3] whose proof is not elementary and where $\overline{T(X)}$ is assumed to be convexly totally bounded instead of the local convexity of the space E. Our proof of Theorem 1 is elementary, but does not work for the proof of [2, Theorem 4.3]; see [9].

2. If X itself is compact, Theorem 1 reduces to Himmelberg [1, Theorem 1], whose proof is based on the Kakutani theorem. If X itself is convex, Theorem 2 reduces to the so-called Himmelberg fixed point theorem [1, Theorem 2], which includes theorems due to Brouwer, Schauder, Tychonoff, Hukuhara, Kakutani, Bohnenblust-Karlin, Fan and Glicksberg. Moreover, there have appeared a number of another generalizations of the Himmelberg theorem; see [6].

3. It is well-known that the Brouwer theorem and the KKM principle are equivalent (proof is now well-known; see [9]). Therefore, all of the results mentioned in the above paragraph are actually equivalent. Moreover, the KKM principle and its open-valued version (Lemma 2) are equivalent via the Brouwer theorem. It is elementary to deduce the KKM principle from Lemma 2.

4. If we assume, in addition that X is closed (in this case X must be convex), the conclusion of Theorem 1 remains valid for multimap $T : X \to 2^E$ satisfying $T(x) \cap X \neq \emptyset$ for $x \in X$. In this case we have not $\overline{T(X)} \subset X$, but we can use the method in the proof of Theorem 1 for $K = \overline{T(X)} \cap X$ with slight modifications. Even more simply, we can apply Theorem 1 to the map $T' : X \to 2^X$ defined by $T'(x) = T(x) \cap X$ for $x \in X$.

Acknowledgement. This work was done while the first author was visiting the Hanoi Institute of Mathematics, Vietnam, in January-February, 1999, and supported in part by Ministry of Education, Korea, Project Number BSRI-98-1413, and by the National Basic Research Program in Natural Sciences, Vietnam.

References

[1] C.J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.

[2] A. Idzik, *Almost fixed point theorems*, Proc. Amer. Math. Soc. **104** (1988), 779–784.

[3] B. Knaster, K. Kuratowski und S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.

[4] S. Park, *A unified approach to generalizations of the KKM type theorems related to acyclic maps*, Numer. Funct. Anal. and Optimiz. **15** (1994), 105–119.

[5] S. Park, *Eighty years of the Brouwer fixed point theorem*, Antipodal Points and Fixed Points (by J.Jaworowski, W.A.Kirk, and S.Park), Lect. Notes Ser. 28, RIM-GARC, Seoul, Nat. Univ., 1995, 55–97.

[6] S. Park, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829.

[7] S. Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. to appear.

[8] S. Park and H. Kim, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl. **209** (1997), 551-571.

[9] S. Park and D.H. Tan, *Remarks on the Schauder-Tychonoff fixed point theorem*, to appear.

[10] M.H. Shih, *Covering properties of convex sets*, Bull. London Math. Soc. **18** (1986), 57–59.