AN ALTERNATIVE PRINCIPLE FOR CONNECTED ORDERED SPACES

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1. Introduction

In [R1], B. Ricceri considered spaces admitting a continuous bijection onto $[0,1]$ (simply, we will call them $[0,1]$-*spaces*) and, based on a new alternative principle for multifunctions involving $[0,1]$-spaces, obtained new mini-max theorems in full generality and transparence. Several further consequences of the principle have been investigated in successive works; see [R2, R3, Ci, CB].

In our previous work [P1], we deduced some fixed point theorems for connected $[0,1]$-spaces from Ricceri's alternative principle. Even though these theorems were consequences of known theorems for an interval $[a,b]$, in general, they seem to be quite new. More general theorems for connected ordered spaces were recently obtained in [P2] with a different method.

In the present paper, we are mainly concerned with connected topological spaces which admit continuous bijections onto a connected ordered spaces with two end points. In Section 2, we deduce a Ricceri type alternative principle and fixed point theorems. Section 3 deals with the incorrect proof of Maćkowiak [M, Theorem 3.1], which was the main tool in [C]. In Section 4, we show that our alternative principle is a generalization of Szabó's theorem [S]. Section 5 deals with the coincidence theorems of Charatonik [C]. In fact, we give new and correct proofs based on our principle and the affirmative answer to Charatonik's question.

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2. A Ricceri type alternative principle

A *multifunction* $T : X \to 2^Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a multifunction $T : X \to 2^Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A multifunction $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c) if for each closed set $B \subset Y$, the set $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^{-1}(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. multifunction T with closed values is closed.

We need the following [H, Theorem 3.1]:

Lemma. *Let $\Gamma : X \multimap Y$ be l.s.c. (or u.s.c.). Suppose that $C \subset X$ is connected and that $\Gamma(x)$ is connected for all $x \in C$. Then the image of C under Γ is connected.*

A linearly ordered set $(X, \leq)$ is called an *ordered space* if it has the order topology whose subbase consists of all sets of the form $\{x \in X : x < s\}$ and $\{x \in X : x > s\}$ for $s \in X$. Note that an ordered space X is connected iff it is Dedekind complete (that is, every subset of X having an upper bound has a supremum) and whenever $x < y$ in X, then $x < z < y$ for some z in X; for details, see Willard [Wi].

We give some examples of connected ordered space X with two end points:

(a) Connected $[0, 1]$-spaces; that is, connected spaces admitting a continuous bijection onto the unit interval; see [R1], [P1].

(b) An *arc* is a homeomorphic image of the unit interval $[0, 1]$. A *generalized arc* is a continuum (not necessarily metrizable) having exactly two non-cut points. It is well-known that a generalized arc is an arc if and only if it is metrizable. It is known [W] that cut points are used to define a natural order on a connected set, and that any generalized arc admits a natural linear order which will be denoted by $\leq$.

A connected ordered space with two end points a, b with $a < b$ will be denoted by $[a, b]$.

Motivated by Ricceri [R1], we obtain the following alternative principle:

Theorem 1. *Let X be a connected topological space, and Y a topological space admitting a continuous bijection onto a connected ordered space $[a,b]$. Let $F, G : X \to 2^Y$ be maps satisfying one of the following two conditions:*

(i) *F, G are l.s.c. with connected values;*

(ii) *F, G are u.s.c. with compact connected values.*

Under such assumptions, at least one of the following two assertions does hold:

(a) *$F(X) \neq Y$ and $G(X) \neq Y$.*

(b) *There exists some $\tilde{x} \in X$ such that $F(\tilde{x}) \cap G(\tilde{x}) \neq \emptyset$.*

Proof. Let φ be a continuous bijection from Y onto a connected ordered space $[a,b]$. Suppose that $F(x) \cap G(x) = \emptyset$ for all $x \in X$ and $F(X) = Y$. Let

$$H(x) = \varphi(F(x)) \times \varphi(G(x)) \quad \text{for} \quad x \in X.$$

We claim that $H(X)$ is connected.

Case (i). The multifunctions $\varphi \circ F$ and $\varphi \circ G$ are l.s.c. with connected values. It is easily checked that H is l.s.c.; see [B].

Case (ii). The multifunctions $\varphi \circ F$ and $\varphi \circ G$ are u.s.c. with compact connected values. Then H is u.s.c. with compact connected values; here, the compact-valuedness of $\varphi \circ F$ and $\varphi \circ G$ are essential in order to assure the u.s.c. of H; see [B].

Then, by Lemma, $H(X)$ is connected in any case.

Now we show that $H(X)$ is also disconnected: Let $A, B \subset [a,b] \times [a,b]$ such that

$$A := \{(s,t) : s < t\} \quad \text{and} \quad B := \{(s,t) : s > t\}.$$

Then A and B are open and disjoint, and we clearly have

$$H(X) \subset A \cup B.$$

Choose $x_a, x_b \in X$ such that $\varphi^{-1}(a) \in F(x_a)$ and $\varphi^{-1}(b) \in F(x_b)$. Pick $y_a \in G(x_a)$ and $y_b \in G(x_b)$. Then we have $\varphi(y_a) > a$; for, otherwise, we would have $F(x_a) \cap G(x_a) \neq \emptyset$. Likewise, we have $\varphi(y_b) < b$.

Consequently,

$$(a, \varphi(y_a)) \in A \cap H(X) \text{ and } (b, \varphi(y_b)) \in B \cap H(X).$$

Then $H(X)$ becomes the union of two disjoint nonempty open subsets $A \cap H(X)$ and $B \cap H(X)$. This contradicts the connectivity of $H(X)$.

Remarks. 1. We followed the proof of Ricceri [R1, Theorem 2.1], which is the case of Theorem 1 for a $[0, 1]$-space Y.

2. However, our result is already known by Ricceri, since he noted that his result [R1, Theorem 2.1] is still true if $[0, 1]$ is replaced by any topological space T having the following property: there are two open (or closed) subsets A, B of $T \times T$ and two points $s_0, t_0 \in T$ such that $(T \times T)\backslash\Delta \subset A \cup B$, $A \cap B \subset \Delta$, $\{s_0\} \times (T\backslash\{s_0\}) \subset A$, and $\{t_0\} \times (T\backslash\{t_0\}) \subset B$, where Δ is the diagonal of $T \times T$.

From Theorem 1, we have the following:

Theorem 2. *Let X be a topological space, Y a topological space admitting a continuous bijection onto a connected ordered space $[a, b]$, and S a connected subset of $X \times Y$. Moreover, let $\Phi : X \to 2^Y$ be a multifunction which is either l.s.c. with connected values, or u.s.c. with compact connected values. Then, at least one of the following holds:*

- (a_1) *$p_Y(S) \neq Y$ and $\Phi(p_X(S)) \neq Y$, where p_X and p_Y are projections from $X \times Y$ to X and Y, resp.*
- (a_2) *There exists some $(\tilde{x}, \tilde{y}) \in S$ such that $\tilde{y} \in \Phi(\tilde{x})$.*

Proof. We may assume $S \neq \emptyset$. Define $F, G : S \to 2^Y$ by

$$F(x, y) = \{y\} \quad \text{and} \quad G(x, y) = \Phi(x) \quad \text{for } (x, y) \in S.$$

Then the conclusion follows from Theorem 1.

Remark. For a $[0, 1]$-space Y Theorem 2 reduces to Ricceri [R1, Theorem 2.2].

From Theorems 1 and 2, we deduce the following fixed point theorem on multifunctions:

Theorem 3. *Let X be a connected ordered space with two end points. Then a multifunction $F : X \to 2^X$ has a fixed point if it satisfies one of the following conditions:*

(I) *F has connected graph.*

(II) *F is l.s.c. with connected values.*

(III) *F is u.s.c. with compact connected values.*

(IV) *$F(x)$ is connected and $F^{-1}(y)$ is open for each $x, y \in X$.*

(V) *F is a closed compact multifunction with connected values.*

Proof. (I)-(V) are all simple consequences of Theorems 1 and 2 as follows:

(I) Theorem 2 with $X = Y,\ S = \mathrm{Gr}(F)$, and $\Phi = id_X$, the identity map on X.

(II) Theorem 1(i) with $X = Y$ and $G = id_X$.

(III) Theorem 1(ii) with $X = Y$ and $G = id_X$.

(IV) Since $F^{-1}(y)$ is open for each $y \in X$, F is l.s.c. Indeed, for each open set $\Omega \subset X$, we have

$$F^{-1}(\Omega) = \{x \in X : F(x) \cap \Omega \neq \emptyset\} = \bigcup_{y \in \Omega} F^{-1}(y)$$

is open. Therefore, (IV) follows from (II).

(V) It is well-known that a closed compact multifunction is u.s.c. with compact values. Therefore, (V) follows from (III).

Remark. Theorem 3 was given in [P2] with different proof.

3. On a coincidence theorem of Maćkowiak

In 1981, Maćkowiak [M] introduduced componentwise continuous (c.c.) multifunctions and used them to obtain some fixed point theorems which generalize most known fixed point theorems for trees, dendroids, and λ-dendroids. Moreover, he obtained a coincidence theorem [M, Theorem 3.1] for two c.c. multifunctions from a connected Hausdorff space X into a generalized arc I.

For Hausdorff compact spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be *componentwise continuous* (c.c.) [M] if $x = \lim\{x_\sigma\}$ implies that

(a) $\mathrm{Ls}\{C_\alpha\} \cap F(x) \neq \emptyset$, where C_α is a component of $F(x_\sigma)$ for each σ [$\mathrm{Ls}\{C_\sigma\}$ is the superior limit of the net $\{C_\sigma\}$]; and

(b) every component of $F(x)$ intersects $\mathrm{Ls}\{F(x_\sigma)\}$.

In [M], many examples of c.c. multifunctions were given and, among them are

(1) lower semicontinuous (l.s.c.) multifunctions with connected values, and

(2) upper semicontinuous (u.s.c.) multifunctions with closed connected values.

Recall that a *generalized arc* is a continuum which has exactly two non-cut points.

Theorem M. [M, Theorem 3.1] *Let c.c. multifunctions F and G map a connected space X into a generalized arc I. Assume that one of the following conditions holds:*

(i) *F is a surjection with connected values.*

(ii) *F and G are both surjections.*

Then there is an $x \in X$ such that $F(x) \cap G(x) \neq \emptyset$.

On the other hand, in an unpublished work of the present author, he tried to apply Theorem M to obtain a common generalization of Theorem M and Ricceri's alternative principle [R1]. However, an excellent referee of that work realized and informed the present author that, unfortunately, the proof of Theorem M is wrong. The referee wrote as follows:

Indeed, using the same notations as in [M], take:

$$X = I = [0,1],$$
$$F(x) = \{x\},$$
$$G(x) = \begin{cases} \{x\} & \text{if } x \in [0,1) \\ [0,1] & \text{if } x = 1. \end{cases}$$

Observe that both the multifunctions F, G are upper semicontinuous, with compact and connected values. So, they are c.c. Moreover, F and G are both surjections. Hence, all the assumptions of Theorem M are satisfied. Now, consider the set A introduced in the proof. Namely,

$$A = \{x \in [0,1] : G(x) \subset [x,1]\}.$$

In the proof, it is claimed that A is closed. In the present case, this is not true. Indeed, we clearly have

$$A = [0,1).$$

Knowing that A is closed is absolutely necessary in the approach adopted in [M]. Consequently, Theorem M, in the absence of a correct proof, should be considered as a conjecture.

4. Generalizations of Szabó's theorem

In 1994, Szabó [S] obtained a coincidence theorem for two continuous multifunctions from a connected space into the set of closed connected subsets of $[0,1]$. Further, he raised a problem how one can generalize his result, in particular for other space than $[0,1]$. In 1997, the present author [P1] and Charatonik [C] gave affirmative solutions to the problem, independently. However, in [C], it was noted that [M, Theorem 3.1] is much stronger than Szabó's theorem.

Let X be a connected topological space. According to Szabó [S], let $K[0,1]$ denote the set of closed connected subsets of $[0,1]$, and a function $F : X \to K[0,1]$ is said to be continuous if each $F(x)$ for $x \in X$ is $[f_0(x), f_1(x)]$ where $f_0, f_1 : X \to [0,1]$ are continuous.

The following is due to Szabó [S]:

Theorem S. *Let $F, G : X \to K[0,1]$ be continuous functions and assume that*

$$\bigcup_{x \in X} F(x) = [0,1].$$

Then there exists $x_0 \in X$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.

Moreover, Szabó [S] raised the following:

Problem S. *How can we generalize Theorem S for other spaces instead of* $[0,1]$*?*

In our previous work [P1], we showed that Theorem S is still true if $[0,1]$ is replaced by any space T in Remark 2 of Theorem 1.

More precisely, Theorem 1 generalizes Theorem S and is an affirmative solution of Problem S.

5. On coincidence theorems of Charatonik

Since any generalized arc admits a natural linear order $\leq$, it can be denoted by $[a,b]$.

Given a generalized arc $[a,b]$, Charatonik [C] denoted by $K[a,b]$ the set of closed connected subsets of $[a,b]$. Thus each nondegenerate element of $K[a,b]$ is a generalized arc $[c,d]$ with $a \leq c < d \leq b$. Adopting the definition from Szabó [S] to this more general case, Charatonik [C] defined the following:

Definition C. Let a generalized arc $[a,b]$ be fixed. A multifunction $F : X \to K[a,b]$ is said to be *continuous* provided that if $F(x) = [f_0(x), f_1(x)]$, then the function $f_0 : X \to [a,b]$ and $f_1 : X \to [a,b]$ are continuous.

Then Charatonik [C] obtained the following:

Statement C. Let X be a space and $Y = [a, b]$ a generalized arc. Then a multifunction $F : X \to K[a, b] \subset 2^Y$ is continuous (u.s.c. and l.s.c.) if and only if it is continuous in the sense of Definition S.

Note that in the above argument, a generalized arc can be replaced by any connected ordered space with two end points.

Analyzing carefully assumptions of Theorem S, Charatonic [C] showed that, in the light of Statment C, the theorem can be reformulated as follows:

Theorem C. *Let X and Y be topological spaces and let multifunctions $F, G : X \to 2^Y$ be given. Assume that*

(1) *X is connected;*
(2) *Y is a generalized arc (or more generally, a connected ordered space with two end points);*
(3) *Y is metrizable;*
(4) *F is l.s.c.;*
(5) *F is u.s.c.;*
(6) *F has compact values;*
(7) *F has connected values;*
(8) *F is surjective;*
(9) *G is l.s.c.;*
(10) *G is u.s.c.;*
(11) *G has compact values;*
(12) *G has connected values.*

Then

(13) *There exists $x_0 \in X$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.*

Note that we replaced the closedness in (6) and (11) by compactness.

Charatonik [C] formulated the following as a possible generalization of Theorem S (or, equivalently, of Theorem C) for Hausdorff spaces X:

Proposition C_1. *If X is a Hausdorff space, then in Theorem C assumptions (3), (5), (6), (10), and (11) can be omitted.*

In the proof of Proposition C_1, the author applied Theorem M. In view of Section 3 of the present paper, the proof can not be complete.

However, without assuming Hausdorffness of X, Proposition C_1 follows immediately from Theorem 1(i).

Another modification of Theorem S is the following in [C].

Proposition C_2. *If X is a Hausdorff space, then in Theorem C assumptions (3), (4), and (9) can be omitted.*

This also follows from Theorem 1(ii) without assuming Hausdorffness of X.

Therefore, we answered affirmatively to the following raised in [C]:

Question C. Can the assumption that the space X is Hausdorff be omitted in Propositions C_1 and C_2?

Note that the Hausdorffness in Propositions C_1 and C_2 came from [M, Theorem 3.1] and is not necessary because our proofs are based on Theorem 1.

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