REMARKS ON TOPOLOGIES OF GENERALIZED CONVEX SPACES

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Abstract. In the KKM theory, by giving finer topologies on the underlying spaces, we can destroy many of artificial terminology. We also show that more simple formulations of some KKM type theorems and some Fan–Browder type fixed point theorems are possible.

1. Introduction

The KKM theory is the study of applications of various equivalent formulations of the classical Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle); see Park [P1-6, PK]. At the beginning, the theory was mainly devoted to study on convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [L], and to spaces having certain families of contractible subsets (simply, C-spaces or H-spaces) by Horvath [H1,2]. Moreover, those spaces are all included to the generalized convex (simply, G-convex) spaces due to the author, and the basic theory on G-convex spaces was extensively studied; for the literature, see [P2,4, PK] and references therein.

One of the equivalent formulations of the KKM theorem is the Fan-Browder fixed point theorem which states that, for a compact convex subset X of a topological vector space, a multimap $T : X \multimap X$ with nonempty convex values and

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open inverse values has a fixed point. In 1977, Tarafdar [Ta] first noted that the open inverse valuedness of T can be replaced by $X = \bigcup_{y\in X} \mathrm{Int}_X T^-(y)$ in the Fan-Browder theorem.

Moreover, in 1987, Kim [K] and Shih and Tan [ST] simultaneously showed that the KKM theorem holds for KKM maps having open values instead of the original closed values. Further, in 1992, Tian [Ti] showed that Fan's generalization of the KKM theorem holds for "transfer closed-valued" KKM maps instead of closed-valued maps. According to Tian, Tarafdar's condition can be said that T^- is transfer open-valued.

Recently, many papers have appeared in the KKM theory devoting to refine or generalize Tian's concepts. Some of those papers are concerned with quite artificial situations by adopting new terminology. Since this kind of terminology seems to be not practical and non-productive, it is desirable to study the true nature of those artificial concepts generalizing Tian's ones. In fact, in this paper, we show that, by giving finer topologies on the underlying spaces, we can destroy many of such artificial terminology. We also show that more simple forms of some KKM type theorems and some Fan–Browder type fixed point theorems are possible. We will give our arguments in the frame of generalized convex spaces as in our recent work [P4].

2. The KKM type theorems

We recall the following in [P4]:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that, for each $A = \{a_0, a_1, \cdots, a_n\} \in \langle D\rangle$, there exist a subset $\Gamma(A) = \Gamma_A$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_A(\Delta_J) \subset \Gamma(\{a_j : j \in J\})$, where $\langle D\rangle$ denotes the set of all nonempty finite subsets of D, Δ_n an n-simplex with vertices $v_0, v_1, \cdots, v_n$, and $\Delta_J =$co$\{v_j : j \in J\}$ the face of Δ_n corresponding to J.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X; \Gamma) := (X, X; \Gamma)$.

There are a large number of examples of G-convex spaces. Typical examples are convex subsets of topological vector spaces, convex spaces in the sense of Lassonde [L], C-spaces (or H-spaces) due to Horvath [H1,2], and many others; see [P2-6, PK].

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$.

The following is well-known:

The KKM Principle. *Let D be the set of vertices of an n-simplex Δ_n and $F : D \multimap \Delta_n$ be a KKM map (that is, $\operatorname{co} N \subset F(N)$ for each $N \in \langle D \rangle$) with closed [resp. open] values. Then $\bigcap_{z \in D} F(z) \neq \emptyset$.*

The following is a KKM theorem for G-convex spaces [P4,6], and its proof is given here for completeness:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a mutimap such that*

(1.1) F has closed [resp. open] values; and

(1.2) F is a KKM map.

Then $\{F(z)\}_{z \in D}$ has the finite intersection property.

Furthermore, if

(1.3) $\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,

then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$$

Proof. Let $N = \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$. Then there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that, for any $0 \le i_0 < i_1 < \cdots < i_k \le n$, we have

$$\phi_N(\operatorname{co}\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_N(\Delta_n).$$

Since F is a KKM map, it follows that

$$\begin{aligned}\mathrm{co}\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\} &\subset \phi_N^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_N(\Delta_n)) \\ &\subset \bigcup_{j=0}^{k} \phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n)).\end{aligned}$$

Since $F(a_{i_j}) \cap \phi_N(\Delta_n)$ is closed [resp. open] in the compact subset $\phi_N(\Delta_n)$ of Γ_N, $\phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n))$ is closed [resp. open] in Δ_n. Note that $v_i \mapsto \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n))$ is a KKM map on $\{v_0, v_1, \ldots, v_n\}$. Hence, by the KKM principle, we have

$$\bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n)) \neq \emptyset,$$

which readily implies $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$. Therefore, $\{F(z)\}_{z \in D}$ has the finite intersection property. Now, by (1.3), the final conclusion follows.

By closely examining the above proof, the values $F(z)$ might be *compactly closed* [resp. *compactly open*] in X; see [P4]. Moreover, the proof works if we assume that, for each $z \in D$ and $N \in \langle D \rangle$, $F(z) \cap \Gamma_N$ is closed [resp. open] in Γ_N, instead of (1.1). This is usually said that F has *finitely closed* [resp. *finitely open*] values; see [T].

It should be noted that the conclusions of the KKM type theorems are certain set-theoretical intersection properties of functional values of the KKM maps, and hence, it is possible to change the topologies of the underlying spaces without loss of generality. For this purpose, we recall the following:

A topological space X is called a *compactly generated space* (or a *k-space*) iff the following condition holds:

(K) $A \subset X$ is open [resp. closed] iff $A \cap K$ is open [resp. closed] in K for each compact set K in X (that is, iff A is compactly open [resp. compactly closed]).

There are a lot of examples of compactly generated spaces; see [W].

A G-convex space $(X, D; \Gamma)$ is said to have *finitely generated topology* iff the following condition holds:

(F) $A \subset X$ is open [resp. closed] iff $A \cap \Gamma_N$ is open [resp. closed] in Γ_N for each $N \in \langle D \rangle$ (that is, iff A is finitely open [resp. finitely closed]).

A convex subset of a topological vector space with the finite topology has the finitely generated topology, and any convex space in the sense of Lassonde [L] has the finitely generated topology. Note that Tan [T, p.4152, Definitions 1.3 and 1.6] misunderstood the author's intention.

Recall that condition (1.1) of Theorem 1 can be replaced by the following:

$(1.1)'$ *F has compactly closed* [*resp. compactly open*] *values; or*

$(1.1)''$ *F has finitely closed* [*resp. finitely open*] *values.*

However, by adopting the compactly generated topology on X or the finitely generated topology on $(X, D; \Gamma)$, it is easily seen that $(1.1)'$ or $(1.1)''$ simply implies (1.1).

For any given subset A of a topological space X, Ding [D1,2] defined *the compact closure* and the *compact interior* of A, denoted by $\mathrm{ccl}(A)$ and $\mathrm{cint}(A)$, as

$\mathrm{ccl}(A) = \bigcap\{B \subset X : A \subset B$ and B is compactly closed in $X\}$, and
$\mathrm{cint}(A) = \bigcup\{B \subset X : B \subset A$ and B is compactly open in $X\}$, respectively.

However, these are simply the closure and the interior of A whenever X has the compactly generated topology.

Similarly, "*finitely* metrically closed sets" in Kirk et al. [KSY] could be simplified.

For a multimap $F : D \multimap X$, we define a multimap $\overline{F} : D \multimap X$ by $\overline{F}(z) := \overline{F(z)}$ for all $z \in D$, where $\overline{}$ denotes the closure operator.

From the closed version of Theorem 1, we can deduce the following equivalent formulation:

Theorem 2. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*

(2.1) $\bigcap_{z \in D} \overline{F(z)} = \bigcap_{z \in D} F(z)$ [*F is transfer closed-valued*];

(2.2) *$\overline{F}$ is a KKM map; and*

(2.3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$.*

Then we have $\bigcap_{z\in D} F(z) \neq \emptyset$.

Proof. The map $\overline{F} : D \multimap X$ is a KKM map with closed values. Hence, by Theorem 1, we have $\bigcap_{z\in D} \overline{F(z)} \neq \emptyset$. Then, by (2.1), we have $\bigcap_{z\in D} F(z) = \bigcap_{z\in D} \overline{F(z)} \neq \emptyset$.

Note that (1.1) $\Longrightarrow$ (2.1) and not conversely, and that Theorem 2 reduces to Theorem 1 if F has closed values. However, Theorems 1 and 2 are equivalent.

Theorem 2 originates from Tian [Ti, Theorem 2]. He defined as follows: Let X be a set and Y a topological space. A map $G : X \multimap Y$ is said to be *transfer open* [resp. *transfer closed*]*-valued* if for each $x \in X$, $y \in G(x)$ [resp. $y \notin G(x)$] implies that there exists a point $x' \in X$ such that $y \in \operatorname{Int} G(x')$ [resp. $y \notin \overline{G(x')}$].

Note that G is transfer open-valued iff $G(X) := \bigcup_{x\in X} G(x) = \bigcup_{x\in X} \operatorname{Int} G(x)$; and G is transfer closed-valued iff $\bigcap_{x\in X} G(x) = \bigcap_{x\in X} \overline{G(x)}$.

More refined concept than the above definition is recently due to Wu and Zhang [WZ] as follows: A family $\{D_\alpha : \alpha \in I\}$ of some subsets of a topological space X is called *closed* [resp. *open*] *transfer complete* if for each $x \in X$ such that $x \notin D_{\alpha_0}$ [resp. $x \in D_{\alpha_0}$] for some $\alpha_0 \in I$, there exists $\alpha' \in I$ such that $x \notin \overline{D_{\alpha'}}$ [resp. $x \in \operatorname{Int} D_{\alpha'}$]. Obviously, if $\{D_\alpha : \alpha \in I\}$ is a family of some closed [resp. open] subset of X, then it is closed [resp. open] transfer complete. A multimap $T : Y \multimap X$ is said to be *transfer closed-valued* if the family $\{T(y) : y \in Y\}$ is closed transfer complete.

Moreover, Ding [D1,2] extended the above definition as follows: Let X and Y be two topological spaces. A map $G : X \multimap Y$ is said to be *transfer compactly open-valued* [resp. *transfer compactly closed-valued*] on X if for $x \in X$ and for each nonempty compact subset K of Y, $y \in G(x) \cap K$ [resp. $y \notin G(x) \cap K$] implies that there exists a point $x' \in X$ such that $y \in \operatorname{Int}_K(G(x') \cap K)$ [resp. $y \notin \operatorname{cl}_K(G(x') \cap K)$].

However, this concept reduces to that of transfer open-valued [resp. transfer closed-valued] if we give Y the compactly generated topology.

3. The Fan-Browder type theorems

From Theorem 1, we deduce the following new form of the Fan–Browder fixed point theorem:

Theorem 3. *Let* $(X, D; \Gamma)$ *be a G-convex space and* $S : X \multimap D$, $T : X \multimap X$ *maps such that*

(3.1) *for each* $x \in X$, $M \in \langle S(x) \rangle$ *implies* $\Gamma_M \subset T(x)$;

(3.2) $S^-(z)$ *is open [resp. closed] for each* $z \in D$; *and*

(3.3) $X = \bigcup \{S^-(z) : z \in N\}$ *for some* $N \in \langle D \rangle$.

Then T *has a fixed point* $x_* \in X$; *that is,* $x_* \in T(x_*)$.

Proof. Let $F : D \multimap X$ be defined by $F(z) := X \backslash S^-(z)$ for $z \in D$. Then each $F(z)$ is closed [resp. open] by (3.2). Note that $\bigcap_{z \in N} F(z) = X \backslash \bigcup_{z \in N} S^-(z) = X \backslash X = \emptyset$ by (3.3) and hence, $\{F(z)\}_{z \in D}$ does not have the finite intersection property. Therefore, by Theorem 1, F can not be a KKM map and hence, there exist an $N \in \langle D \rangle$ and an $x_* \in \Gamma_N$ such that $x_* \notin F(N)$. Since $x_* \notin F(z)$ or $x_* \in S^-(z)$ for all $z \in N$, we have $z \in S(x_*)$ for all $z \in N$ and hence $N \in \langle S(x_*) \rangle$. Therefore, by (3.1), we have $\Gamma_N \subset T(x_*)$. Since $x_* \in \Gamma_N$, we have $x_* \in T(x_*)$. This completes our proof.

Theorem 3′. *Let* $(X, D; \Gamma)$ *be a G-convex space and* $S : X \multimap D$, $T : X \multimap X$ *maps such that*

(3.1)′ *for each* $x \in X$, $M \in \langle S(x) \rangle$ *implies* $\Gamma_M \subset T(x)$; *and*

(3.3)′ $X = \bigcup \{\mathrm{Int}\, S^-(z) : z \in N\}$ *for some* $N \in \langle D \rangle$.

Then T *has a fixed point.*

Proof. In Theorem 3, replace S^- by $\mathrm{Int}\, S^-$. Then $M \in \langle (\mathrm{Int}\, S^-)^-(x) \rangle \subset \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$ by (3.1)′, and hence (3.1) follows. Note that (3.3)′ implies (3.3). Therefore, the conclusion follows from Theorem 3.

Theorem 3′ is known as the Fan–Browder fixed point theorem when $X = D$ is a compact convex subset of a topological vector space.

Note that condition $(3.3)'$ originates from Tarafdar [Ta]. It is clear that $(3.3)'$ implies that S^- is transfer open-valued, and not conversely.

From Theorem $3'$, we have the following equivalent form:

Theorem $3''$. *If we replace* $(3.3)'$ *in Theorem* $3'$ *by*

$(3.3)''$ $\bigcup\{S^-(z) : z \in D\} = \bigcup\{\operatorname{Int} S^-(z) : z \in D\}$ [S^- *is transfer open-valued*]

and if X *is compact, then either*

(a) *there exists an* $x_0 \in X$ *such that* $S(x_0) = \emptyset$*; or*

(b) T *has a fixed point* $x_1 \in X$.

Proof. Suppose that for any $x \in X$, there exists a $z \in D$ such that $z \in S(x)$ or $x \in S^-(z)$. Then $(3.3)''$ implies $X = \bigcup\{\operatorname{Int} S^-(z) : z \in D\}$. If X is compact, this implies $(3.3)'$, and hence (b) holds by Theorem $3'$.

In case when $X = D$, the point x_0 in (a) is called a *maximal element* of S.

Wu and Shen [WS] defined as follows: For topological spaces X and Y, a map $G : X \multimap Y$ is said to have *the local intersection property* on X if for each $x \in X$ with $G(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x in X such that $\bigcap_{z \in N(x)} G(z) \neq \emptyset$. Note that, if G has open inverse images $G^-(y)$, then G has the local intersection property.

It is easily seen that if G has nonempty values, then G has local intersection property iff G^- is transfer open-valued. This is said that G is *transfer open inversed valued* in [KSY].

Ding [D2] refined the above concept as follows: A map $G : X \multimap Y$ is said to have the *compactly local intersection property* on X if $G|_K$ has the local intersection property for any nonempty compact subset K of X.

As we mentioned several times, Ding's concept reduces to usual one if we adopt the compactly generated topology.

4. The equilibrium theorems

In order to discuss another terminology, we consider the following generalized form of Tan, Yu, and Yuan [TY, Theorem 2.1]:

Theorem 4. *Let* $(X, D; \Gamma)$ *be a compact* G*-convex space and* $f : X \times D \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ *a function such that for each* $r \in \mathbb{R}$,

(4.1) $\bigcup_{z\in D}\{x \in X : f(x,z) > r\} = \bigcup_{z\in D} \mathrm{Int}\{x \in X : f(x,z) > r\}$; *and*

(4.2) *the map* $z \longmapsto \overline{\{x \in X : f(x,z) \leq r\}}$ *is KKM on* D.

Then there exists $x^* \in X$ *such that* $f(x^*, z) \leq r$ *for all* $z \in D$.

Proof. Let $F : D \multimap X$ be defined by $F(z) := \{x \in X : f(x,z) \leq r\}$ for $z \in D$. Then $\overline{F} : D \multimap X$ is a KKM map on a compact G-convex space $(X, D; \Gamma)$. Note that, (4.1) is equivalent to $\bigcap_{z\in D} F(z) = \bigcap_{z\in D} \overline{F(z)}$. Therefore, by Theorem 2, we have $x^* \in \bigcap_{z\in D} F(z)$. This completes our proof.

Note that (4.1) simply tells that F is transfer closed-valued.

This fact is defined by Wu and Zhang [WZ] as follows: For a set X, a topological space Y, and $f : X \times Y \to \mathbb{R}$, $f(x,y)$ is said to be W*-lower* [resp. W*-upper*] *semicontinuous* in y if for each $y \in Y$ and each $r \in \mathbb{R}$ with $\{x \in X : f(x,y) > r\} \neq \emptyset$ [resp. $\{x \in X : f(x,y) < r\} \neq \emptyset$], there exists $x' \in X$ such that $y \in \mathrm{Int}\{z \in Y : f(x',z) > r\}$ [resp. $y \in \mathrm{Int}\{z \in Y : f(x',z) > r\}$].

Obviously, if $f(x,y)$ is lower (upper) semicontinuous in y, then $f(x,y)$ is W-lower (upper) semicontinuous in y.

Moreover, Tian [Ti] and Wu and Li [WL] defined the following: For a topological space X and a set Y, a function $f : X \times Y \to \mathbb{R}$ is said to be *transfer lower* [resp. *upper*] *semicontinuous* about x on X if for each $r \in \mathbb{R}$, and each $x \in X$ and $y \in Y$, $f(x,y) > r$ [resp. $f(x,y) < r$] implies that there exists an open neighborhood $N(x)$ of x and a point $y' \in Y$ such that $f(z,y') > r$ [resp. $f(z,y') < r$] for all $z \in N(x)$.

Wu and Li [WL] also noted that: If f is lower [resp. upper] semicontinuous in x, then f is transfer lower [resp. upper] semicontinuous about x on X. If a function $f : X \times Y \to \mathbb{R}$ is transfer lower [resp. upper] semicontinuous about x on X, then the marginal function $x \mapsto \sup_{y \in Y} f(x,y)$ is lower semicontinuous [resp. $x \mapsto \inf_{y \in Y} f(x,y)$ is upper semicontinuous]. The converse is not true.

Further, Ding [D2] obtained more general terminology as follows: Let X and Y be two topological spaces, $\lambda \in \mathbb{R}$, and $T : X \multimap Y$ a multimap. A function $\phi : X \times Y \times X \to \mathbb{R}$ is said to be λ-*transfer compactly upper semicontinuous* in x with respect to T if for each compact subset K of X, $\{z \in X : \sup_{y \in T(x)} \phi(x,y,z) < \lambda\} \neq \emptyset$ implies that there exists an open neighborhood $N(x)$ of x and a point $z' \in X$ such that $\sup_{y \in T(u)} \phi(u,y,z') < \lambda$ for all $u \in N(x) \cap K$. If we define a multimap $G : X \multimap Y$ by $G(x) = \{z \in X : \sup_{y \in T(x)} \phi(x,y,z) < \lambda\}$, then G has the compactly local intersection property if and only if $\phi(x,y,z)$ is λ-transfer compactly upper semicontinuous in x with respect to T.

Therefore, ϕ is λ-transfer compactly upper semicontinuous in x with respect to T if and only if

$$\bigcup_{x \in X} \{z \in X : \sup_{y \in T(x)} \phi(x,y,z) < \lambda\} = \bigcup_{x \in X} \mathrm{Int}\{z \in X : \sup_{y \in T(x)} \phi(x,y,z) < \lambda\},$$

or if and only if the map $F : X \multimap Y$ defined by

$$F(x) = \{z \in X : \sup_{y \in T(x)} \phi(x,y,z) \geq \lambda\} \quad \text{for } x \in X$$

is transfer closed-valued, whenever we adopt the compactly generated topology on X.

It should be emphasized that, sometimes, the analytical expression is more informative than the artificial terminology.

Note that Theorem 4 is an example of equilibrium theorems. There are lots of generalized forms of Theorem 4.

5. For G-convex spaces $(X \supset D; \Gamma)$

For a G-convex space $(X \supset D; \Gamma)$, a subset Y of X is called a *G-convex subspace* of $(X \supset D; \Gamma)$ if $(Y, Y \cap D; \Gamma')$ is a G-convex space where $\Gamma'_A := \Gamma_A \cap Y$ for $A \in \langle Y \cap D \rangle$.

For H-spaces, many authors called weakly H-convex subsets for our G-convex subspaces. We will not list any of such examples.

For a G-convex space $(X \supset D; \Gamma)$, we have another form of the KKM theorem with a more general coercivity (or compactness) condition:

Theorem 5. *Let $(X \supset D; \Gamma)$ be a G-convex space, K a nonempty compact subset of X, and $F : D \multimap X$ a multimap such that*

(5.1) $\bigcap_{z\in D} F(z) = \bigcap_{z\in D} \overline{F(z)}$;

(5.2) $\overline{F}$ *is a KKM map; and*

(5.3) *for each $N \in \langle D \rangle$, there exists a compact G-convex subspace L_N of X containing N such that*

$$L_N \cap \bigcap \{\overline{F(z)} : z \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap\{F(z) : z \in D\} \neq \emptyset$.

Proof. Suppose that $K \cap \bigcap\{F(z) : z \in D\} = K \cap \bigcap\{\overline{F(z)} : z \in D\} = \emptyset$; that is, $K \subset \bigcup\{X \setminus \overline{F(z)} : z \in D\}$. Let $G : D \multimap X$ be defined by $G(z) := X \setminus \overline{F(z)}$ for $z \in D$. Then we have $K \subset \bigcup_{z\in D} G(z)$. Since K is compact, there exists an $N \in \langle D \rangle$ such that $K \subset \bigcup_{z\in N} G(z)$. Let L_N be the compact G-convex subspace of X in (5.3). Define $F' : L_N \cap D \multimap L_N$ by $F'(z) := F(z) \cap L_N$ for $z \in L_N \cap D$. Then $A \in \langle L_N \cap D \rangle$ implies $\Gamma'_A := \Gamma_A \cap L_N \subset \overline{F}(A) \cap L_N = \overline{F'}(A)$ by (5.2); and hence $\overline{F'} : L_N \cap D \multimap L_N$ is a KKM map on $(L_N, L_N \cap D; \Gamma')$ with closed values. By Theorem 1, $\{\overline{F'(z)} : z \in L_N \cap D\}$ has the finite intersection property and $\bigcap\{\overline{F'(z)} : z \in L_N \cap D\} \neq \emptyset$. For any $y \in \bigcap\{\overline{F'(z)} : z \in L_N \cap D\}$, we have $y \in K$ by (5.3). However, since $y \in K \subset \bigcup\{X \setminus \overline{F(z)} : z \in N\}$, we have $y \notin \overline{F(z)}$

for some $z \in N \subset L_N \cap D$. This is a contradiction. Therefore, we must have $K \cap \bigcap\{F(z) : z \in D\} \neq \emptyset$. This completes our proof.

Examples 1. For a convex subset X of a Hausdorff topological vector space and $\emptyset \neq D \subset X$, Tian [Ti, Theorems 2 and 3] obtained particular forms of Theorem 5 adopting one of the following conditions instead of (5.3):

(2c) there is a nonempty subset D_0 of D such that the intersection $\bigcap_{z \in D_0} \overline{F(z)}$ is compact and D_0 is contained in a compact convex subset of X.

(3c) there is a nonempty subset $D_0 \subset D$ such that for each $x \in X \backslash D_0$ there exists a point $z \in D_0$ with $x \notin \overline{F(z)}$ and D_0 is contained in a compact convex subset of X.

Tian [Ti] applied his theorems to obtain the Ky Fan type minimax inequality, existence of maximal elements, existence of price equilibrium, and solutions of the complementarity problems.

2. Ding [D1, Corollaries 3.1 and 3.2] restated Tian's results by adopting terminology like as transfer compactly closed and compact closure, and claimed that his results improve corresponding ones of Tian.

3. Chang et al. [CLW, Lemma 2.2] obtained a particular form of Theorem 5 for an H-space $(X; \Gamma)$.

4. Lin and Park [LP, Lemma 1] obtained Theorem 5 for the case $X = D$.

As in our previous works [P4, PK], Theorem 5 can have more than a dozen equivalent formulations. Here we give only one example as follows.

The following popular form of generalized Fan–Browder fixed point theorem is equivalent to Theorem 5:

Theorem 6. *Let $(X \supset D; \Gamma)$ be a G-convex space, K a nonempty compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that*

(6.1) *for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$;*

(6.2) *$K \subset \bigcup\{\mathrm{Int}\, S^-(z) : z \in D\}$; and*

(6.3) *for each $N \in \langle D \rangle$, there exists a compact G-convex subspace L_N of X containing N such that*

$$L_N \backslash K \subset \bigcup\{\mathrm{Int}\, S^-(z) : z \in L_N \cap D\}.$$

Then T has a fixed point.

Examples 1. There are more general formulations of Theorem 6 mainly due to the author, and many particular forms are still happening. Moreover, conditions (5.3) and (6.3) are first due to S. Y. Chang [C] as generalizations of Ky Fan's original conditions, and have been adopted by the present author from 1992. However, still many scholars are using particular forms of (5.3) and (6.3).

In the following, we list only a few particular forms of Theorem 6 which were obtained very recently.

2. Tan and Zhang [TZ, Theorems 3.1-3.3] obtained particular forms of Theorem 6 for the case $X = D = K$ is compact. Moreover, they had to assume the Hausdorffness of X contrary to their claim.

3. Let X be a convex subset of a topological vector space and $G : X \multimap X$ a map such that $G^{-}(y)$ is compactly open for each $y \in X$. Ding and Yuan [DY, Theorem 2.2] showed that there exists $x \in X$ such that $x \in \operatorname{co} G(x)$ under the assumtion that

(c) *there exist a nonempty compact convex subset X_0 and a nonempty compact subset K of X such that, for each $x \in X\backslash K$, we have that* $\operatorname{co}(X_0 \cup \{x\}) \cap \operatorname{co} G(x) \neq \emptyset$.

We show that their result follows from Theorem 6 under the assumption that $G(x) \neq \emptyset$ for all $x \in X$. Let $X = D$ and $S = T = \operatorname{co} G$. Then $T(x)$ is nonempty convex for each $x \in X$, and $S^{-}(y) = (\operatorname{co} G)^{-}(y)$ is easily seen to be open as $G^{-}(y)$ is open for each $y \in X$. This shows the conditions (6.1) and (6.2) hold.

We claim that (c) implies condition (6.3). For any $N \in \langle X \rangle$, let $L_N := \operatorname{co}(X_0 \cup N)$. Then L_N is a compact convex subset of X as so is X_0. For any $x \in L_N \backslash K$, by (c) there exists a $z \in \operatorname{co}(X_0 \cup \{x\}) \cap \operatorname{co} G(x)$ and hence $z \in L_N = L_N \cap D$ and $z \in \operatorname{co} G(x) = S(x)$. Therefore, $x \in S^{-}(z) = (\operatorname{co} G)^{-}(z) = \operatorname{Int}(\operatorname{co} G)^{-}(z) = \operatorname{Int} S^{-}(z)$. Therefore, (6.3) holds.

Now, by Theorem 6, there exists an $x \in X$ such that $x \in T(x) = \operatorname{co} G(x)$.

Finally, we obtain a more simple formulation of Theorem 6 as follows:

Theorem 6′. *Let* $(X \supset D; \Gamma)$ *be a* G*-convex space,* K *a nonempty subset of* X*, and* $S : X \multimap D$*,* $T : X \multimap X$ *multimaps. Suppose that*

$(6.1)'$ *for each* $x \in X$*,* $M \in \langle S(x) \rangle$ *implies* $\Gamma_M \subset T(x)$*;*

$(6.2)'$ $K \subset \bigcup\{\operatorname{Int} S^-(z) : z \in N\}$ *for some* $N \in \langle D \rangle$*; and*

$(6.3)'$ *there exists a* G*-convex subspace* L_N *of* X *containing* N *such that*

$$L_N \backslash K \subset \bigcup\{\operatorname{Int} S^-(z) : z \in M\}$$

for some $M \in \langle L_N \cap D \rangle$*.*

Then T *has a fixed point in* L_N*.*

Proof. Note that $K \subset \bigcup\{\operatorname{Int} S^-(z) : z \in N\}$ for some $N \in \langle D \rangle$. For this N, we have a G-convex subspace $(L_N, L_N \cap D; \Gamma')$ as in $(6.3)'$. In this subspace L_N,

$$L_N = (L_N \backslash K) \cup (L_N \cap K) \subset \bigcup\{\operatorname{Int} S^-(z) : z \in M \cup N\},$$

where $M \cup N \in \langle L_N \cap D \rangle$. Therefore, by Theorem 3, the conclusion follows.

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