THE KNASTER-KURATOWSKI-MAZURKIEWICZ THEOREM AND ALMOST FIXED POINTS

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1. Introduction

It is well-known that the three classical results – the Brouwer fixed point theorem, the Sperner lemma, and the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem – are mutually equivalent in the sense that each one can be deduced from another with or without aid of some minor results. Earlier applications of the Sperner lemma and the KKM theorem to various results closely related to Euclidean spaces or n-simplexes or n-balls were presented in [1,2]; see also [13].

Especially, one of the earlier proofs of the Brouwer theorem was given by the KKM theorem in [10]. Recently, the present author and Do Hong Tan [14] gave a simple proof of a generalization of the Schauder-Tychonoff type fixed point theorem for compact maps in locally convex Hausdorff topological vector spaces, by

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directly using the KKM theorem. Subsequently, in [15], elementary proofs of generalizations of the Himmelberg-Idzik type fixed point theorem for convex-valued upper semicontinuous multimaps were obtained from the "open" valued version of the KKM theorem due to Kim [9] and Shih-Tan [16].

On the other hand, in 1957, Alexandroff-Pasynkoff [3] gave an elementary proof of the essentiality of the identity map of the boundary of a simplex by using a variant of the KKM theorem. Consequently, a simple proof of the Brouwer theorem could be given by using their theorem; see [13]. Moreover, in 1990, Lassonde [11] suggested an "open" version of the Alexandroff-Pasynkoff theorem.

In the present paper, we first deduce a generalization of the Alexandroff-Pasynkoff theorem by using the KKM theorem for the "closed" and "open" valued cases and note that these two theorems are actually equivalent. And then, the KKM theorem is applied to our main result, which concerns with the existence of almost fixed points of lower [resp. upper] semicontinuous multimaps. Our new result is general enough to include properly previous results on almost fixed points due to Ky Fan [7] and Lassonde [11], and fixed point theorems due to Park and Tan [14, 15], Himmelberg [8], and many others. Finally, applying our main result, we give a partial solution to a conjecture raised by Ben-El-Mechaiekh [4, 5]

2. The KKM and Alexandroff-Pasynkoff theorems

From the KKM theorem and its open version, we have immediately the following form as in Fan [6]:

THEOREM 1. *Let X be a subset of a topological vector space, D a nonempty subset of X such that* $\operatorname{co} D \subset X$, *and* $F : D \multimap X$ *a multimap with closed* [*resp. open*] *values in* X. *If*

$$\operatorname{co} A \subset F(A) \tag{1}$$

for every nonempty finite subset A of D, then the family $\{F(x)\}_{x\in D}$ has the finite intersection property.

The open version of the KKM theorem was due to Kim [9] and Shih-Tan [16], and later, Lassonde [11] showed that the closed and open versions of the KKM theorem can be derived from each other.

From Theorem 1, we have the following generalization of the Alexandroff-Pasynkoff theorem [3]:

THEOREM 2. *Let X be a subset of a topological vector space, $\{A_i\}_{i=0}^n$ a family of $(n+1)$ closed [resp. open] subsets covering X, and $\{x_i\}_{i=0}^n$ a family of $(n+1)$ points of X such that* $\mathrm{co}\{x_i\}_{i=0}^n \subset X$ *and*

$$\mathrm{co}\{x_0, \cdots, \widehat{x_i}, \cdots, x_n\} \subset A_i \quad \textit{for each } i = 0, 1, \cdots, n.$$

Then

$$\bigcap_{i=0}^{n} A_i \neq \emptyset.$$

PROOF. Let $D := \{x_i\}_{i=0}^n$ and let $C_0 := \mathrm{co}\{x_0, x_1, \cdots, x_{n-1}\} \subset A_n$ and $C_i := \mathrm{co}\{x_0, \cdots, \widehat{x_{i-1}}, \cdots, x_n\} \subset A_{i-1}$ for $1 \leq i \leq n$. Let $F : D \multimap X$ be a map defined by $F(x_0) = A_n$ and $F(x_i) = A_{i-1}$ for $1 \leq i \leq n$. Now we show that F satisfies the requirement of Theorem 1. Note that

$$\mathrm{co}\{x_0, x_1, \cdots, x_n\} \subset X = \bigcup_{i=0}^{n} A_i = F(D).$$

Moreover, for any proper subset

$$\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\}, \; (0 \leq k < n, \; 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$$

of D, we immediately have

$$\mathrm{co}\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset C_{i_j} \subset A_{i_j - 1} = F(x_{i_j})$$

for some j, $0 \le j \le k$, [with the convention $i_j = 0$ iff $i_j - 1 \equiv n$] and hence

$$\mathrm{co}\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset \bigcup_{j=0}^{k} F(x_{i_j}).$$

Consequently, condition (1) is satisfied. Now, the conclusion follows from Theorem 1. □

It is well-known that the Alexandroff-Pasynkoff theorem implies the Brouwer theorem (e.g., see [13]). Therefore, Theorem 2 is also equivalent to the KKM theorem.

3. Almost fixed point theorems

From Theorem 1, in this section, we deduce a very general almost fixed point theorem and some of its direct applications.

A nonempty subset Y of a topological vector space E is said to be almost convex [8] if for any neighborhood V of the origin 0 of E and for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of Y, there exists a finite subset $\{z_1, z_2, \cdots, z_n\}$ of Y, such that $z_i - y_i \in V$ for each $i = 1, \cdots, n$, and $\mathrm{co}\{z_1, z_2, \cdots, z_n\} \subset Y$.

The following almost fixed point theorem is our main result in this paper:

THEOREM 3. *Let X be a subset of a Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(y)$ is convex for all $y \in Y$. If there is a precompact subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$, then for any convex neighborhood U of the origin 0 of E, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

PROOF. There exists a symmetric open neighborhood V of 0 such that $\overline{V} + \overline{V} \subset U$. Since K is precompact in E, there exists a finite subset $\{x_0, x_1, \cdots, x_n\} \subset K$ such that $K \subset \bigcup_{i=0}^{n}(x_i + V)$. Moreover, since Y is almost convex and dense in

X, there exists a finite subset $D = \{y_0, y_1, \cdots, y_n\}$ of Y such that $y_i - x_i \in V$ for each $i = 0, 1, \cdots, n$, and $Z := \mathrm{co}\{y_0, y_1, \cdots, y_n\} \subset Y$.

If T is lower semicontinuous, for each i, let

$$F(y_i) := \{z \in Z : T(z) \cap (x_i + V) = \emptyset\},$$

which is closed in Z. Moreover we have

$$\bigcap_{i=0}^{n} F(y_i) = \{z \in Z : T(z) \cap \bigcup_{i=0}^{n} (x_i + V) = \emptyset\} = \emptyset$$

since $\emptyset \neq T(z) \cap K \subset T(z) \cap \bigcup_{i=0}^{n} (x_i + V)$ for each $z \in Y$.

If T is upper semicontinuous, for each i, let

$$F(y_i) := \{z \in Z : T(z) \cap (x_i + \overline{V}) = \emptyset\},$$

which is open in Z. Moreover we have

$$\bigcap_{i=0}^{n} F(y_i) = \emptyset$$

as in the above.

Now we apply Theorem 1 replacing (X, D) by $(Z, \{y_i\}_{i=0}^{n})$. Since the conclusion of Theorem 1 does not hold, in any case, condition (1) is violated. Hence, there exist a subset $N := \{y_{i_0}, \cdots, y_{i_k}\} \in \langle D \rangle$ and an $x_U \in \mathrm{co}\, N \subset Y$ such that $x_U \notin F(N)$ or $T(x_U) \cap (x_{i_j} + \overline{V}) \neq \emptyset$ for all $j = 0, 1, \cdots, k$. Note that

$$x_{i_j} + \overline{V} = x_{i_j} - y_{i_j} + y_{i_j} + \overline{V} \subset y_{i_j} + V + \overline{V} \subset y_{i_j} + U. \tag{2}$$

Let L be the subspace of E generated by D and

$$M := \{y \in L : T(x_U) \cap (y + U) \neq \emptyset\}.$$

From (2) we get $T(x_U) \cap (y_{i_j} + U) \neq \emptyset$ and hence $y_{i_j} \in M$ for all $j = 0, 1, \cdots, k$. Since $L, T(x_U)$, and U are all convex, it is easily checked that M is convex. Therefore, $x_U \in M$ and, by definition of M, we get $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes our proof. $\square$

In case $X = Y$, Theorem 3 reduces to the following:

COROLLARY 4. *Let X be a convex subset of a Hausdorff topological vector space E. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(x)$ is convex for each $x \in X$. If there is a precompact subset K of X such that $T(x) \cap K \neq \emptyset$ for each $x \in X$, then for every convex neighborhood U of the origin 0 of E, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Ky Fan [7, Theorem 7] obtained Corollary 4 for a locally convex Hausdorff topological vector space E and for a lower semicontinuous multimap $T : X \multimap E$. For a single-valued map $f : X \to X$, Fan noted that Corollary 4 might be regarded as a generalization of the Tychonoff fixed point theorem to noncompact (or precompact) convex sets; see Corollary 5 below.

Lassonde [5, Théorème 4] obtained Corollary 4 for a compact upper semicontinuous map $T : X \multimap X$ having nonempty convex values.

From Theorem 3, we have the following fixed point theorem:

COROLLARY 5. *Let X be a subset of a locally convex Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap X$ be a compact upper semicontinuous multimap with closed values such that $T(y)$ is nonempty convex for all $y \in Y$. Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

PROOF. By Theorem 3, for each neighborhood U of 0, there exist $x_U,\ y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(X)$ is relatively compact, we may assume that the net $\{y_U\}$ converges to some $x_0 \in K$. Since E is Hausdorff, the net $\{x_U\}$ also converges to x_0. Because T is upper semicontinuous with closed values, the graph of T is closed in $X \times T(X)$ and hence we have $x_0 \in T(x_0)$. This completes our proof. □

Corollary 5 is recently due to the author and Do Hong Tan [15] and extends the Himmelberg-Idzik theorem and many other fixed point results in the analytical fixed point theory.

From Theorem 3, we have the following almost fixed point result:

Corollary 6. *Let X be a subset of a Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap E$ be a multimap such that* (1) $T^{-}(z)$ *is open for each* $z \in E$; *and* (2) $T(y)$ *is convex for for each* $y \in Y$. *If there is a precompact subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$, then for any convex neighborhood U of the origin* 0 *of E, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Proof. Since T is lower semicontinuous, Corollary 6 follows immediately from Theorem 3. □

In case $X = Y$, Corollary 6 reduces to the following:

Corollary 7. *Let X be a convex subset of a Hausdorff topological vector space E, and $T : X \multimap X$ be a multimap such that* (1) $T(x)$ *is nonempty and convex for each* $x \in X$; (2) $T^{-1}(y)$ *is open for each* $y \in X$; *and* (3) $T(X)$ *is contained in a compact subset K of X. Then for any convex neighborhood U of the origin* 0 *of E, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Ben-El-Mechaiekh [1, 2] obtained that, if E is further assumed to be locally convex in Corollary 7, then T has a fixed point; and conjectured that, under the hypotheses of Corollary 7, T would have a fixed point. This conjecture is not resolved yet; for partial solutions, see [12]. However, Corollary 7 is a new partial solution.

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