PERGAMON

Mathematical and Computer Modelling 34 (2001) 947–954

MATHEMATICAL AND COMPUTER MODELLING

www.elsevier.nl/locate/mcm

Fixed Points and Quasi-Equilibrium Problems†

S. PARK*
Department of Mathematics
Seoul National University
Seoul 151-742, Korea
shpark@math.snu.ac.kr

Abstract—A fixed-point theorem for compact acyclic maps defined on convex subsets of not-necessarily locally convex topological vector spaces is applied to the existence of solutions of quasi-equilibrium problems. Such existence theorems extend known ones which were used for unified approaches to quasi-variational inequalities in [1–5] and others. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Admissible topological vector space (t.v.s.), Acyclic map, Compact map, Closed map, Upper semicontinuous (u.s.c.), Lower semicontinuous (l.s.c.), Quasi-equilibrium problem, Symmetric quasi-equilibrium theorem.

1. INTRODUCTION

We obtain a fixed-point theorem for compact acyclic maps defined on convex subsets of not-necessarily locally convex topological vector spaces. Our theorem generalizes a large number of historically well-known results. Moreover, for locally convex topological vector spaces, our new theorem reduces to the earlier results of the author [6–9], which were applied to the existence theorems of solutions of variational or quasi-variational inequalities and other problems. See [1–5].

In the present paper, our new theorem is applied to the existence of solutions of quasi-equilibrium problems in admissible topological vector spaces (in the sense of Klee [10]). Such existence theorems extend known ones which were used for unified approaches to quasi-variational inequalities or other problems in [1–5,11].

2. A FIXED-POINT THEOREM

A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^-(y)$ if and only if $y \in T(x)$. For topological spaces X and Y, a map $T : X \multimap Y$

A preliminary version of this paper was given as a part of an invited talk at the International Mathematics Conference, Mathematical Society of Republic of China, held on December 12–16, 1996. The author would like to express his sincere gratitude to the hospitality of its organizers.
*Supported in part by the Brain Korea 21 Project.
†This article originally appeared in Volume 32, Issue 11–13 of this journal and due to an error on the part of the printers it is being reprinted here.
PII of original article: S0895-7177(00)00204-1

PII: S0895-7177(01)00111-X

is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x,y) : x \in X,\, y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y. A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X: *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^-(B)$ is open: and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. map T with closed values is closed.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any convex or star-shaped subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \multimap Y$ is said to be *acyclic* if it is u.s.c. with compact acyclic values.

Recall that an extended real-valued function $g : X \to \bar{\mathbf{R}}$ on a topological space X is *lower* (respectively, *upper*) *semicontinuous* (l.s.c.) (respectively, u.s.c.), if $\{x \in X : g(x) > r\}$ (respectively, $\{x \in X : g(x) < r\}$) is open for each $r \in \bar{\mathbf{R}}$. If X is a convex set in a vector space, then $g : X \to \bar{\mathbf{R}}$ is *quasi-concave* (respectively, *quasi-convex*), if $\{x \in X : g(x) > r\}$ (respectively, $\{x \in X : g(x) < r\}$) is convex for each $r \in \bar{\mathbf{R}}$.

Throughout this paper, all topological spaces are assumed to be Hausdorff, t.v.s. means topological vector spaces, and co denotes the convex hull.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of [10]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Note that every nonempty convex subset of a locally convex t.v.s. is admissible. See [12]. Other examples of admissible t.v.s. are ℓ^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on [0,1], the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. or any compact convex locally convex subset of a t.v.s. is admissible. For details, see [10,13] and references therein.

The following fixed-point theorem is the basis of our arguments in this paper.

THEOREM 1. *Let E be a t.v.s. and X an admissible convex subset of E. Then, any compact acyclic map $F : X \multimap X$ has a fixed point $x \in X$; that is, $x \in F(x)$.*

PROOF. Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin 0 of E. Since F is closed and compact, it is sufficient to show that for any $V \in \mathcal{V}$, there exists an $x_V \in X$ such that $(x_V + V) \cap F(x_V) \neq \emptyset$.

Since $\overline{F(X)}$ is a compact subset of the admissible subset X, there exist a continuous map $h : \overline{F(X)} \to X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{F(X)}$ and $h(\overline{F(X)}) \subset L \cap X$. Let $M := h(\overline{F(X)})$. Then, M is a compact subset of L, and hence, $P := \mathrm{co}\, M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{F(X)} \to P$ and $F|_P : P \multimap \overline{F(X)}$. Since h and $F|_P$ are acyclic maps, it is well known that their composition $h \circ (F|_P)$ has a fixed point $x_V \in P$; that is, $x_V \in h \circ F(x_V)$, and hence, $x_V = h(y)$ for some $y \in F(x_V)$. Since $y - h(y) \in V$, we have $y \in h(y) + V = x_V + V$. Therefore, $(x_V + V) \cap F(x_V) \neq \emptyset$. This completes our proof. ∎

Generalized forms of Theorem 1 were due to the author [14]. However, a simple proof of Theorem 1 is given here for completeness. Theorem 1 generalizes many well-known fixed-point theorems even when E is locally convex and F has convex values such as the works of Brouwer, Schauder, Tychonoff, Mazur, Kakutani, Hukuhara, Bohnenblust and Karlin, Fan, Glicksberg, Rhee, Himmelberg, and others. For the related results and the literature, see [6–9,15].

Note that the convexity of X in Theorem 1 is not essential. In fact, X can be any subset of E which is homeomorphic to an admissible convex subset of another t.v.s. E_1. However, throughout this paper, we state all results for admissible convex sets for simplicity.

3. QUASI-EQUILIBRIUM THEOREMS

By an *equilibrium problem*, Blum and Oettli [16] understood the problem of finding

$$\bar{x} \in K, \quad \text{such that } f(\bar{x}, y) \geq 0, \qquad \text{for all } y \in K,$$

where K is a given set and $f : K \times K \to \mathbf{R}$ is a given function with $f(x,x) \geq 0$ for all $x \in K$. They observed that this problem contains as special cases, for instance, optimization problems, problems of Nash equilibria, complementarity problems, fixed-point problems, variational inequalities, and many others. Some variations or generalizations of this problem can be possible. See [11] and references therein.

From Theorem 1, we deduce the following existence theorem for a quasi-equilibrium problem (in the sense of [11]).

THEOREM 2. *Let X and Y be admissible convex subsets of t.v.s. E and F, respectively, $S : X \multimap X$ a compact closed map, $T : X \multimap Y$ a compact acyclic map, and $\phi : X \times Y \times X \to \bar{\mathbf{R}}$ an u.s.c. function. Suppose that*

(i) *the function $m : X \times Y \to \bar{\mathbf{R}}$ defined by*

$$m(x,y) = \max_{s \in S(x)} \phi(x,y,s), \qquad \text{for } (x,y) \in X \times Y$$

is l.s.c., and

(ii) *for each $(x,y) \in X \times Y$, the set*

$$M(x,y) = \{u \in S(x) : \phi(x,y,u) = m(x,y)\}$$

is acyclic.

Then, there exists an $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\bar{x} \in S(\bar{x}), \quad \bar{y} \in T(\bar{x}), \quad \text{and} \quad \phi(\bar{x},\bar{y},\bar{x}) \geq \phi(\bar{x},\bar{y},s), \qquad \text{for all } s \in S(\bar{x}).$$

PROOF. We first show that $M : X \times Y \multimap X$ is a compact acyclic map. Note that for each $(x,y) \in X \times Y$, $M(x,y)$ is nonempty since $\phi(x,y,\cdot)$ is u.s.c. on the compact set $S(x)$. We show that $\mathrm{Gr}(M)$ is closed in $X \times Y \times X$. In fact, let $(x_\alpha, y_\alpha, u_\alpha) \in \mathrm{Gr}(M)$ and $(x_\alpha, y_\alpha, u_\alpha) \to (x,y,u) \in X \times Y \times X$. Then,

$$\phi(x,y,u) \geq \overline{\lim_\alpha}\, \phi(x_\alpha, y_\alpha, u_\alpha) = \overline{\lim_\alpha}\, m(x_\alpha, y_\alpha) \geq \underline{\lim_\alpha}\, m(x_\alpha, y_\alpha) \geq m(x,y).$$

Since $\mathrm{Gr}(S)$ is closed in $X \times X$, $u_\alpha \in S(x_\alpha)$ implies $u \in S(x)$. Hence, $(x,y,u) \in \mathrm{Gr}(M)$. Moreover, each $M(x,y)$ is acyclic by (ii). Since S is compact and $M(X \times Y) \subset S(X)$, M is also compact. Therefore, M is a compact acyclic map.

Now, we define a map $G : X \times Y \multimap X \times Y$ by

$$G(x,y) = M(x,y) \times T(x), \qquad \text{for } (x,y) \in X \times Y.$$

Then G is an acyclic map. In fact, since M and T are compact-valued and u.s.c., their product G is also compact-valued and u.s.c. Note that each $G(x,y)$ is acyclic since the product of two acyclic sets is acyclic by the Künneth theorem. Moreover, G is compact because $G(X \times Y) \subset \overline{S(X)} \times \overline{T(X)} \subset X \times Y$. Since X and Y are admissible, it is easily checked that $X \times Y$ is admissible in the t.v.s. $E \times F$. Therefore, by Theorem 1, G has a fixed point $(\bar{x}, \bar{y}) \in X \times Y$. Since

$$(\bar{x}, \bar{y}) \in G(\bar{x}, \bar{y}) = M(\bar{x}, \bar{y}) \times T(\bar{x}) \subset S(\bar{x}) \times T(\bar{x}),$$

we have $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$. Moreover, since $\bar{x} \in M(\bar{x}, \bar{y})$, we have

$$\phi(\bar{x}, \bar{y}, \bar{x}) = \max_{s \in S(x)} \phi(\bar{x}, \bar{y}, s).$$

This completes our proof. ∎

REMARK 1. If S and ϕ are continuous in Theorem 2, condition (i) holds automatically. In fact, the function $\phi : (X \times Y) \times X \to \bar{\mathbf{R}}$ is continuous, and the map $S' : X \times Y \multimap X \times Y$ defined by

$$S'(x, y) = S(x) \times \{y\}$$

is continuous with compact values. Therefore, by Berge's theorem [17, Theorem VI.3.2], the marginal function $m : X \times Y \to \bar{\mathbf{R}}$ defined by

$$m(x, y) = \sup \{\phi(x, y, u) : (u, y) \in S'(x, y)\} = \max_{u \in S(x)} \phi(x, y, u)$$

is continuous.

REMARK 2. If $\phi = 0$, Y is a singleton, and T is a constant map, then Theorem 2 reduces to Theorem 1.

REMARK 3. A particular version of Theorem 2 for Euclidean spaces was obtained by Cubiotti and Yao [2, Theorem 3.4], from which [2, Theorem 3.5 and Corollary 3.6] follows.

If Y is a singleton and T is a constant map, then Theorem 2 reduces to the following corollary.

COROLLARY 1. *Let X be an admissible convex subset of a t.v.s. E, $S : X \multimap X$ a compact closed map, and $\phi : X \times X \to \bar{\mathbf{R}}$ an u.s.c. function. Suppose that*

(1) *the function $m : X \to \bar{\mathbf{R}}$ defined by*

$$m(x) = \max_{s \in S(x)} \phi(x, s), \qquad \text{for } x \in X$$

is l.s.c., and

(2) *for each $x \in X$, the set*

$$\{u \in S(x) : \phi(x, y) = m(x)\}$$

is acyclic.

Then, there exists an $\bar{x} \in X$ such that

$$\bar{x} \in S(\bar{x}) \qquad \text{and} \qquad \phi(\bar{x}, \bar{x}) = m(\bar{x}).$$

REMARK 4. When E is locally convex, Corollary 1 reduces to [4, Theorem 2], which extends earlier works of Takahashi, Im and Kim, and Kaczynski and Zeidan. For the literature, see [4].

REMARK 5. A related result to Corollary 1 is also given in [5].

The inward set $I_X(x)$ of X at $x \in E$ is defined as

$$I_X(x) = x + \bigcup_{r > 0} r(X - x),$$

and $\bar{I}_X(x)$ denotes its closure.

From Theorem 2, we have the following theorem.

THEOREM 3. *Under the hypothesis of Theorem 2, if*

(iii) $\phi(x,y,x) \le 0$ *holds for all* $(x,y) \in \mathrm{Gr}(T)$,

then there exists an $(\bar{x},\bar{y}) \in X \times Y$ *such that*

$$\bar{x} \in S(\bar{x}), \quad \bar{y} \in T(\bar{x}), \quad \phi(\bar{x},\bar{y},x) \le 0, \qquad \text{for all } x \in S(\bar{x}).$$

Further, if $\phi : X \times Y \times E \to \bar{\mathbf{R}}$ *is an u.s.c. function such that* $\phi(x,y,x) = 0$ *and* $\phi(x,y,\cdot)$ *is concave and l.s.c. for each* $(x,y) \in X \times Y$, *and if* $S(\bar{x})$ *is convex, then*

$$\phi(\bar{x},\bar{y},x) \le 0, \qquad \text{for all } x \in \bar{I}_{S(\bar{x})}(\bar{x}).$$

PROOF. The first part is clear from

$$0 \ge \phi(\bar{x},\bar{y},\bar{x}) = \max_{s \in S(\bar{x})} \phi(\bar{x},\bar{y},s),$$

which is a consequence of Theorem 2 and condition (iii).

For the second part, since $x \mapsto \phi(\bar{x},\bar{y},x)$ is l.s.c., it is sufficient to show that

$$\phi(\bar{x},\bar{y},x) \le 0, \qquad \text{for all } x \in I_{S(\bar{x})}(\bar{x}) \setminus S(\bar{x}).$$

For any $x \in I_{S(\bar{x})}(\bar{x}) \setminus S(\bar{x})$, there exist $z \in S(\bar{x})$ and $r > 0$ such that $x = \bar{x} + r(z - \bar{x})$. Then, we must have $r > 1$; otherwise, $x = (1-r)\bar{x} + rz \in S(\bar{x})$ since $0 < r \le 1$ and $S(\bar{x})$ is convex. Suppose $\phi(\bar{x},\bar{y},x) > 0$. Since $r > 1$ and

$$\frac{1}{r}x + \left(1 + \frac{1}{r}\right)\bar{x} = z \in S(\bar{x}),$$

we have

$$\phi(\bar{x},\bar{y},z) = \phi\left(\bar{x},\bar{y},\frac{1}{r}x + \left(1 - \frac{1}{r}\right)\bar{x}\right) \ge \frac{1}{r}\phi(\bar{x},\bar{y},x) + \left(1 - \frac{1}{r}\right)\phi(\bar{x},\bar{y},\bar{x}) = \frac{1}{r}\phi(\bar{x},\bar{y},x) > 0,$$

by the concavity of $x \mapsto \phi(\bar{x},\bar{y},x)$ and $\phi(\bar{x},\bar{y},\bar{x}) = 0$. This is a contradiction. Therefore, we have $\phi(\bar{x},\bar{y},x) \le 0$. This completes our proof. ∎

REMARK 6. In [1], from a version of Theorem 3 for locally convex t.v.s., the authors deduced more than a score of known variational or quasi-variational inequality theorems. Using our Theorem 3, those theorems can be shown to hold for not-necessarily locally convex spaces.

In the remainder of this section, assume that the topological dual E^* is admissible and the pairing $\langle \cdot, \cdot \rangle$ on $E^* \times E$ is continuous. For example,

(1) E is a normed vector space, or
(2) E is locally convex and E^* is equipped with the strong topology $\delta(E^*, E)$ or the topology of uniform convergence on compact subsets of E.

COROLLARY 2. *Let* X *be an admissible convex subset of a t.v.s.* E, $S : X \multimap X$ *and* $T : X \multimap E^*$ *compact acyclic maps, and* Y *an admissible convex subset of* E^* *containing* $\overline{T(X)}$. *Let* $\phi : X \times Y \times X \to \bar{\mathbf{R}}$ *be u.s.c. such that conditions (i)–(iii) hold. Then, the conclusion of Theorem 3 holds.*

REMARK 7. Even in case E is locally convex, Corollary 2 improves [1, Theorem 3].

REMARK 8. If E is locally convex and S and ϕ are continuous, then Corollary 2 reduces to [3, Theorem 1], which was used to obtain generalized quasi-variational inequality theorems.

We give an application of Corollary 2 to a variational inequality extending Yao [18, Theorem 3.1].

COROLLARY 3. *Let K be an admissible compact convex subset of a t.v.s. E, and $f : K \to E^*$, $g : K \to E$ continuous maps. Suppose that*

$$\left\{ u \in K : \langle f(x), g(u) \rangle = \min_{v \in K} \langle f(x), g(v) \rangle \right\}$$

is acyclic for each $x \in K$. Then, there exists an $\bar{x} \in K$ such that

$$\langle f(\bar{x}), g(u) - g(\bar{x}) \rangle \geq 0, \qquad \text{for all } u \in K.$$

PROOF. We use Corollary 2 with $X = K$. Let $S(x) = K$ for $x \in K$, $T = -f$, $Y = E^*$, and $\phi(x, z, u) = \langle -f(x), g(u) - g(x) \rangle$ on $K \times E^* \times K$. Then, $\phi : K \times E^* \times K \to \mathbf{R}$ is continuous. Since $S : K \multimap K$ is a compact continuous map and ϕ is continuous, condition (i) of Theorem 2 holds. Note that $\phi(x, z, x) = 0$ for all $(x, z, x) \in K \times E^* \times K$, and hence, condition (iii) of Theorem 3 holds. Moreover, for each $(x, z) \in K \times E^*$,

$$M(x, z) = \left\{ u \in K : \phi(x, z, u) = \max_{s \in K} \phi(x, z, s) \right\} = \left\{ u \in K : \langle f(x), g(u) \rangle = \min_{s \in K} \langle f(x), g(s) \rangle \right\}$$

is acyclic by assumption, and hence, condition (ii) of Theorem 2 holds. Therefore, by Corollary 2, there exists an $(\bar{x}, \bar{z}) \in K \times E^*$ such that

$$\bar{x} \in S(\bar{x}) = K, \quad \bar{z} = -f(\bar{x}), \quad \phi(\bar{x}, \bar{z}, u) \leq 0, \qquad \text{for all } u \in S(\bar{x}) = K,$$

which implies the conclusion. ∎

4. SYMMETRIC QUASI-EQUILIBRIUM THEOREMS

From Theorem 1, we have the following symmetric quasi-equilibrium theorem (in the sense of [11]).

THEOREM 4. *Let X and Y be admissible convex subsets of t.v.s. E and F, respectively, $S : X \times Y \multimap X$ and $T : X \times Y \multimap Y$ compact acyclic maps, and $f, g : X \times Y \to \bar{\mathbf{R}}$ l.s.c. functions such that*

(i) *the functions*

$$F(x, y) = \min\{f(\xi, y) : \xi \in S(x, y)\},$$
$$G(x, y) = \min\{g(x, \eta) : \eta \in T(x, y)\}$$

are u.s.c. on $X \times Y$, and

(ii) *for each $(x, y) \in X \times Y$, the sets*

$$A(x, y) = \{\xi \in S(x, y) : f(\xi, y) = F(x, y)\},$$
$$B(x, y) = \{\eta \in T(x, y) : g(x, \eta) = G(x, y)\}$$

are acyclic.

Then, there exists an $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \quad f(x, \bar{y}) \geq f(\bar{x}, \bar{y}), \qquad \text{for all } x \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}), \quad g(\bar{x}, y) \geq g(\bar{x}, \bar{y}), \qquad \text{for all } y \in T(\bar{x}, \bar{y}).$$

PROOF. As in the proof of Theorem 2, the maps $A : X \times Y \multimap X$ and $B : X \times Y \multimap Y$ are compact acyclic maps. Therefore, the map $A \times B : X \times Y \multimap X \times Y$ defined by $(A \times B)(x, y) = A(x, y) \times B(x, y)$ is a compact acyclic map. Since $X \times Y$ is an admissible convex subset of $E \times F$, by Theorem 1, $A \times B$ has a fixed point $(\bar{x}, \bar{y}) \in (A \times B)(\bar{x}, \bar{y})$, which satisfies the conclusion. ∎

REMARK 9. If S and T are convex-valued and if $f(\cdot, y)$ and $g(x, \cdot)$ are quasi-convex in Theorem 4, then (ii) holds automatically. In this case, Theorem 4 for locally convex t.v.s. E and F reduces to [11, Theorem 1].

From Theorem 4, we have another quasi-equilibrium result.

COROLLARY 4. *Let X and Y be admissible convex subsets of t.v.s. E and F, respectively, $S : X \multimap X$ a compact continuous map with closed convex values, $T : X \multimap Y$ a compact u.s.c. map with closed convex values, and $f : X \times Y \to \mathbf{R}$ a continuous map such that $f(\cdot, y)$ is quasi-convex for each $y \in Y$. Then, there exists an $(\bar{x}, \bar{y}) \in X \times Y$ such that*

$$\bar{x} \in S(\bar{x}), \quad \bar{y} \in T(\bar{x}), \quad f(x, \bar{y}) \geq f(\bar{x}, \bar{y}), \qquad \text{for all } x \in S(\bar{x}).$$

PROOF. We use Theorem 4 with $g \equiv 0$. Then, we have the following.

(i) The function $F(x) = \min\{f(\xi, y) : \xi \in S(x)\}$ is u.s.c. on X for each $y \in Y$ by Berge's theorem [17] since S and f are l.s.c.
(ii) The set $A(x) = \{\xi \in S(x) : f(\xi, y) = F(x)\}$ is convex for each $(x, y) \in X \times Y$ since $S(x)$ is convex and $f(\cdot, y)$ is quasi-convex.

Therefore, by Theorem 4, there exists an $(\bar{x}, \bar{y}) \in X \times Y$ satisfying the conclusion. ∎

REMARK 10. For locally convex t.v.s. E and F, Corollary 3 reduces to [11, Theorem 2]. Finally, note that Corollary 3 can be used to extend many variational inequalities given in [1].

We give an example of a quite popular type of general variational inequality.

COROLLARY 5. *Let K be an admissible compact convex subset of a t.v.s. E, and $h : K \to E^*$ and $\theta : K \times K \to E$ continuous maps such that*

(1) *$\langle h(x), \theta(x, x) \rangle \geq 0$, for all $x \in K$, and*
(2) *for each $y \in K$, the function $\langle h(y), \theta(\cdot, y) \rangle : K \to \mathbf{R}$ is quasi-convex.*

Then, there exists an $x_0 \in K$ such that

$$\langle h(x_0), \theta(x, x_0) \rangle \geq 0, \qquad \text{for all } x \in K.$$

PROOF. We use Corollary 4 with $X = K = Y$, $S(x) = K$, $T(x) = x$ for each $x \in K$. Let $f : K \times K \to \mathbf{R}$ be defined by $f(x, y) = \langle h(y), \theta(x, y) \rangle$, for $(x, y) \in K \times K$. Then, f is a continuous map such that $f(\cdot, y)$ is quasi-convex for each $y \in Y$ by (2). Then, by Corollary 4, there exists an $(\bar{x}, \bar{y}) \in K \times K$ such that

$$\bar{x} \in S(\bar{x}), \quad \bar{y} \in T(\bar{x}), \quad f(x, \bar{y}) \geq f(\bar{x}, \bar{y}), \qquad \text{for all } x \in S(\bar{x}).$$

Since $S(\bar{x}) = K$ and $\bar{y} = T(\bar{x}) = \bar{x}$, we have

$$\bar{x} \in K, \quad f(x, \bar{x}) \geq f(\bar{x}, \bar{x}), \qquad \text{for all } x \in K,$$

and hence,

$$\bar{x} \in K, \quad \langle h(\bar{x}), \theta(x, \bar{x}) \rangle \geq \langle h(\bar{x}), \theta(\bar{x}, \bar{x}) \rangle \geq 0, \qquad \text{for all } x \in K,$$

by (1). This completes our proof. ∎

REMARK 11. Particular forms of Corollary 5 appear frequently in the literature; see [1].

REFERENCES

1. M.-P. Chen and S. Park, A unified approach to generalized quasi-variational inequalities, *Comm. Appl. Nonlinear Anal.* **4**, 103–118, (1997).
2. P. Cubiotti and J.C. Yao, Necessary and sufficient conditions for the existence of the implicit variational inequality problem, *Appl. Math. Lett.* **10** (1), 83–87, (1997).
3. S. Kum, A generalization of generalized quasi-variational inequalities, *J. Math. Anal. Appl.* **182**, 158–164, (1994).
4. S. Park, Some existence theorems for two variable functions on topological vector spaces, *Kangwon-Kyunggi Math. J.* **3**, 11–16, (1995).
5. S. Park and M.-P. Chen, Generalized quasi-variational inequalities, *Far East J. Math. Sci.* **3**, 185–190, (1995).

6. S. Park, Some coincide theorems on acyclic multifunctions and applications to KKM theory, In *Fixed Point Theory and Applications*, (Edited by K.-K. Tan), pp. 248–277, World Scientific, River Edge, NJ, (1992).
7. S. Park, Acyclic maps, minimax inequalities, and fixed points, *Nonlinear Analysis, TMA* **24**, 1549–1554, (1995).
8. S. Park, Fixed points of acyclic maps on topological vector spaces, In *World Congress of Nonlinear Analysts '92—Proceedings*, (Edited by V. Lakshmikantham), pp. 2171–2177, Walter de Gruyter, Berlin, (1996).
9. S. Park, S.P. Singh and B. Watson, Some fixed point theorems for composites of acyclic maps, *Proc. Amer. Math. Soc.* **121**, 1151–1158, (1994).
10. O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, Univ. of Novi Sad, Novi Sad, (1984).
11. M.A. Noor and W. Oettli, On general nonlinear complementarity problems and quasi-equilibria, *Le Matematiche* **49**, 313–331, (1994).
12. M. Nagumo, Degree of mappings in convex linear topological spaces, *Amer. J. Math.* **73**, 497–511, (1951).
13. H. Weber, Compact convex sets in non-locally-convex linear spaces, *Note di Matematica* **12**, 271–289, (1992).
14. S. Park, A unified fixed point theory of multimaps on topological vector spaces, *J. Korean Math. Soc.* **35**, 803–829, (1998); Corrections, *J. Korean Math. Soc.* **36**, 829–832, (1999).
15. S. Park, Eighty years of the Brouwer fixed point theorem, In *Antipodal Points and Fixed Points*, Lect. Notes Ser. 28, (Edited by J. Jaworowski *et al.*), pp. 55–97, RIM-GARC, Seoul Nat. Univ., (1995).
16. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Students* **63**, 123–145, (1994).
17. C. Berge, *Espaces Topologiques*, Dunod, Paris, (1959).
18. J.-C. Yao, On the general variational inequality, *J. Math. Anal. Appl.* **174**, 550–555, (1993).
19. V. Klee, Leray-Schauder theory without local convexity, *Math. Ann.* **141**, 286–297, (1960).