

# REMARKS ON FIXED POINT THEOREMS IN HYPERCONVEX SPACES

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ABSTRACT. We generalize and improve known fixed point theorems in hyperconvex spaces by applying the selection theorem in [BO] and other results on  $C$ -spaces or  $G$ -convex spaces. In fact, we obtain generalizations of fixed point theorems for nonexpansive multimaps, the Kakutani type maps, and the Fan-Browder type maps. Some additional new observations are also stated for the Caristi-Kirk-Browder type theorems.

## 1. INTRODUCTION

The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi [AP] in 1956. Later, in 1979, independently Sine [S1] and Soardi [So] proved that a bounded hyperconvex space has the fixed point property for nonexpansive maps. Since then many interesting works have appeared for hyperconvex spaces. For the literature, see the end of this paper.

Until recently, the study of hyperconvex spaces concentrated on the relationship with nonexpansive maps. However, Khamsi [K] established the Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem) for hyperconvex spaces and applied it to obtain a Schauder type fixed point theorem. This line of study was followed by Kirk [Ki2], Kirk and Shin [KS], Kim and Shin [KmS], and Park [P4,5]. In particular, the first author obtained extensions or equivalent forms of the KKM theorem, a Fan-Browder type fixed point theorem, and other results for hyperconvex spaces in [P4,5]. Moreover, Kirk, Sims, and Yuan [KSY] established the KKM theorem, its equivalent formulations, fixed point theorems, and their applications for hyperconvex spaces.

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However, most of the above-mentioned works are simple consequences of much more general results. In fact, Horvath [H1-5] initiated study of the KKM theory and fixed point theory for  $C$ -spaces, which are meaningful generalizations of convex spaces or convex subsets of topological vector spaces. Moreover, in [H5], he found that hyperconvex spaces are a particular type of  $C$ -spaces and gave a useful selection theorem on l.s.c. multimaps related to  $C$ -spaces. Recently, this selection theorem was extended by Ben-El-Mechaiekh and Oudadess [BO] following some ideas from the celebrated theory on continuous selections due to Michael. On the other hand, the first author [P1-3, 7-11, PK1-4] initiated study of generalized convex spaces or  $G$ -convex spaces, which properly include the class of  $C$ -spaces and a large number of spaces having particular type of abstract convexity.

The main purpose of the present paper is to generalize and improve known fixed point theorems in hyperconvex spaces by applying some of the established results for  $G$ -convex spaces, especially, the selection theorem in [BO] and other results on  $C$ -spaces. Some additional new observations are also stated.

In Section 3, we obtain a generalization of the fixed point theorem of Sine [S2] for nonexpansive multimaps on hyperconvex spaces.

Section 4 deals with the fixed point theorems for compact Kakutani type maps on hyperconvex spaces. In fact, from a new theorem of Park [P8], we deduce Himmelberg type theorems for  $G$ -convex spaces and apply them to obtain some variants for hyperconvex spaces.

In Section 5, we deduce generalized forms of a Fan-Browder type theorem for compact maps on hyperconvex spaces.

Section 6 deals with a new generalization of the Caristi-Kirk-Browder fixed point theorem [C]. We note that this can be applied to hyperconvex spaces in order to obtain some new results.

## 2. PRELIMINARIES

A metric space  $(H, d)$  is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_{\alpha}, r_{\alpha}) \neq \emptyset$$

for any collection  $\{B(x_{\alpha}, r_{\alpha})\}$  of closed balls in  $H$  for which  $d(x_{\alpha}, x_{\beta}) \leq r_{\alpha} + r_{\beta}$ .

It is known that the space  $\mathbb{C}(E)$  of all continuous real functions on a Stonian space  $E$  (that is, extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space  $\mathbb{C}(E)$  for some Stonian space  $E$ . Therefore,  $(\mathbf{R}^n, \|\cdot\|_{\infty})$ ,  $l^{\infty}$ , and  $L^{\infty}$  are concrete examples of hyperconvex spaces.

Results of Aronszajn and Panitchpakti [AP, Theorem 1'] and Isbell [I, Theorem 1.1] are combined in the following:

**Theorem 2.1.** *A hyperconvex space is complete and (freely) contractible.*

The concepts of  $C$ -spaces,  $LC$ -spaces, and  $LC$ -metric spaces were introduced and extensively studied by Horvath in a sequences of papers [H1-5]:

A  $C$ -space  $(X, \Gamma)$  is a topological space  $X$  with a multimap  $\Gamma : \langle X \rangle \multimap X$  from the set  $\langle X \rangle$  of all nonempty finite subsets of  $X$  into the power set of  $X$  such that

- (1) for each  $A \in \langle X \rangle$ ,  $\Gamma(A) = \Gamma_A$  is  $n$ -connected for all  $n \geq 0$ ; and
- (2) for all  $A, B \in \langle X \rangle$ ,  $A \subset B$  implies  $\Gamma_A \subset \Gamma_B$ .

A nonempty subset  $Y \subset X$  is said to be  $\Gamma$ -convex if  $A \in \langle Y \rangle$  implies  $\Gamma_A \subset Y$ .

A  $C$ -space  $(X, \Gamma)$  is called an  $LC$ -space (or a *locally  $H$ -convex space* [T]) if  $X$  is a Hausdorff uniform space and there exists a basis  $\{V_\lambda\}_{\lambda \in I}$  for the uniform structure such that for each  $\lambda \in I$ ,  $\{x \in X : E \cap V_\lambda[x] \neq \emptyset\}$  is  $\Gamma$ -convex whenever  $E \subset X$  is  $\Gamma$ -convex, where

$$V_\lambda[x] = \{x' \in X : (x, x') \in V_\lambda\}.$$

For example, any nonempty convex subset  $X$  of a locally convex Hausdorff topological vector space is an  $LC$ -space with  $\Gamma_A = \text{co } A$ , the convex hull of  $A \in \langle X \rangle$ .

A triple  $(X, d; \Gamma)$  is called an  $LC$ -metric space whenever  $(X, d)$  is a metric space and  $(X, \Gamma)$  is a  $C$ -space such that open balls are  $\Gamma$ -convex, and any neighborhood  $\{x \in X : d(x, Y) < r\}$  of a  $\Gamma$ -convex set  $Y \subset X$  is also  $\Gamma$ -convex.

Horvath [H5, Theorem 9] obtained the following :

**Theorem 2.2.** *Any hyperconvex space  $H$  is a complete  $LC$ -metric space with*

$$\Gamma_A = \bigcap \{B : B \text{ is a closed ball containing } A\}$$

for each  $A \in \langle H \rangle$ .

Note that  $\Gamma_A$  itself is hyperconvex. From now on, a hyperconvex space  $(H, d; \Gamma)$  is simply denoted by  $H$ .

The following is due to Ben-El-Mechaiekh and Oudadess [BO, Theorem 3]:

**Theorem 2.3.** *Let  $X$  be paracompact,  $(Y, d; \Gamma)$  a complete  $LC$ -metric space,  $Z \subset X$  with  $\dim_X Z \leq 0$ , and  $\Phi : X \multimap Y$  a lower semicontinuous (l.s.c.) multimap with nonempty closed values such that  $\Phi(x)$  is  $\Gamma$ -convex for  $x \notin Z$ . Then  $\Phi$  admits a continuous selection  $f : X \rightarrow Y$  such that  $f(x) \in \Phi(x)$  for  $x \in X$ .*

Motivated by the concepts of  $C$ -spaces and many other abstract convexities, the first author introduced the following:

A *generalized convex space* or a  $G$ -convex space  $(X, D; \Gamma)$  consists of a topological space  $X$  and a nonempty set  $D$  such that, for each  $A \in \langle D \rangle$  with the

cardinality  $|A| = n + 1$ , there exist a subset  $\Gamma(A) \equiv \Gamma_A$  of  $X$  and a continuous function  $\phi_A : \Delta_n \rightarrow \Gamma(A)$  such that  $J \in \langle A \rangle$  implies  $\phi_A(\Delta_J) \subset \Gamma(J)$ . Note that  $\phi_A|_{\Delta_J}$  can be regarded as  $\phi_J$ .

The above definition of  $G$ -convex spaces is a little more general than the one used in our previous works [P1-3, 7-11, PK1-4], where basic theory was extensively developed. Here,  $\Delta_n$  is the standard  $n$ -simplex with vertices  $\{e_i\}_{i=0}^n$ , and  $\Delta_J$  the face of  $\Delta_n$  corresponding to  $J \in \langle A \rangle$ ; that is, if  $A = \{a_0, a_1, \dots, a_n\}$  and  $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$ , then  $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$ . We may write  $(X, \Gamma) = (X, X; \Gamma)$ .

There are a lot of examples of  $G$ -convex spaces:

**Examples 2.1.** If  $X$  is a convex subset of a vector space,  $D \subset X$ , and  $X$  has a topology such that each  $\Gamma_A$  is the convex hull of  $A \in \langle D \rangle$  equipped with the Euclidean topology, then  $(X, D; \Gamma)$  becomes a convex space generalizing the one due to Lassonde. Note that any convex subset of a topological vector space (t.v.s.) is a convex space, but not conversely.

**Examples 2.2.** If  $X = D$  and  $\Gamma_A$  is assumed to be contractible or, more generally, infinitely connected (that is,  $n$ -connected for all  $n \geq 0$ ), and if for each  $A, B \in \langle X \rangle$ ,  $A \subset B$  implies  $\Gamma_A \subset \Gamma_B$ , then  $(X, \Gamma)$  becomes a  $C$ -space (or an  $H$ -space).

**Examples 2.3.** Other major examples of  $G$ -convex spaces are metric spaces with Michael's convex structure, Pasicki's  $S$ -contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, and so on. For the literature, see [PK1-4]. Recently, further examples of  $G$ -convex spaces were given by the first author [P10] as follows:  $L$ -spaces and  $B'$ -simplicial convexity of Ben-El-Mechaiekh et al., continuous images of  $G$ -convex spaces, Verma's or Stachó's generalized  $H$ -spaces, Kulpa's simplicial structures,  $P_{1,1}$ -spaces of Forgo and Joó,  $mc$ -spaces of Llinares, hypeconvex metric spaces due to Aroszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

**Examples 2.4.** Furthermore, any hyperbolic space  $X$  in the sense of Kirk [Ki1] and Reich-Shafrir [RS] is  $G$ -convex space, since the closed convex hull of any  $A \in \langle X \rangle$  is contractible [PK4, p.542]. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product hyperbolic spaces is also hyperbolic; see [RS].

From now on, all topological spaces are assumed to be Hausdorff and  $D \subset X$  for simplicity. For such a  $G$ -convex space  $(X, D; \Gamma)$ , a subset  $Y$  of  $X$  is said to be  $\Gamma$ -convex if for each  $A \in \langle D \rangle$ ,  $A \subset Y$  implies  $\Gamma_A \subset Y$ .

A  $G$ -convex space  $(X, D; \Gamma)$  is called an  $LG$ -space (or a *locally  $G$ -convex space*) if  $(X, \mathcal{U})$  is a uniform space such that  $D$  is dense in  $X$  and if there exists a basis

$\{V_\lambda\}_{\lambda \in I}$  for the uniformity  $\mathcal{U}$  such that for each  $\lambda \in I$ ,  $\{x \in X : C \cap V_\lambda[x] \neq \emptyset\}$  is  $\Gamma$ -convex whenever  $C \subset X$  is  $\Gamma$ -convex, where

$$V_\lambda[x] = \{x' \in X : (x, x') \in V_\lambda\}.$$

For details, see [P8,9].

### 3. NONEXPANSIVE MULTIMAPS

In their pioneering work, Aronszajn and Panitchpakdi [AP] proved that a hyperconvex space is a nonexpansive retract of any metric space in which it is isometrically embedded. Sine [S1] and Soardi [So] showed independently that a nonexpansive map of a bounded hyperconvex space has a fixed point. Both of these works were in the concrete setting of function spaces. Later, Baillon [B] proved the following :

**Theorem 3.1.** *If  $f : H \rightarrow H$  is a nonexpansive map with  $H$  a bounded hyperconvex space, then the fixed point set  $\text{Fix}(f)$  of  $f$  is nonempty and hyperconvex.*

Recall that for bounded closed sets  $A$  and  $B$  in a metric space  $(X, d)$ , we set

$$h_0(A, B) = \sup\{d(x, B) : x \in A\}$$

and then define the Hausdorff metric by

$$h(A, B) = \max\{h_0(A, B), h_0(B, A)\}.$$

A multimap  $M : X \multimap X$  with nonempty bounded closed values is said to be *nonexpansive* in the Hausdorff metric if

$$h(M(x), M(y)) \leq d(x, y) = h(\{x\}, \{y\}) \quad \text{for all } x, y \in X.$$

From Theorems 2.2 and 2.3, we immediately have the following:

**Theorem 3.2.** *Let  $X$  be a metric space,  $Z \subset X$  with  $\dim_X Z \leq 0$ ,  $H$  a hyperconvex space, and  $\Phi : X \multimap H$  l.s.c. with nonempty closed values such that  $\Phi(x)$  is  $\Gamma$ -convex for  $x \notin Z$ . Then  $\Phi$  admits a continuous selection.*

Since a nonexpansive multimap  $M : X \multimap H$  is l.s.c., from Theorem 3.2, we have the following due to Sine [S2, Theorem 1]:

**Corollary 3.3.** *Let  $M : X \multimap H$  be a nonexpansive multimap from a metric space  $X$  into a hyperconvex space  $H$  having  $\Gamma$ -convex values. Then  $M$  admits a nonexpansive single-valued selection.*

Combining Theorems 3.1 and 3.2, we have the following :

**Theorem 3.4.** *Let  $H$  be a bounded hyperconvex space,  $Z \subset H$  with  $\dim_H Z \leq 0$ , and  $M : H \multimap H$  a nonexpansive multimap such that  $M(x)$  is  $\Gamma$ -convex for  $x \notin Z$ . Then  $M$  has a fixed point.*

*Proof.* Since  $M$  is l.s.c., by Theorem 3.2 with  $X = H$ ,  $M$  admits a continuous selection  $f : H \rightarrow H$ . Since  $d(f(x), f(y)) \leq h(M(x), M(y)) \leq d(x, y)$ ,  $f$  is nonexpansive and hence has a fixed point  $x_0 = f(x_0) \in M(x_0)$  by Theorem 3.1. This completes our proof.

From Theorem 3.4, we have the following form of Sine [S2, Corollary 2]:

**Corollary 3.5.** *Let  $M : H \multimap H$  be a nonexpansive multimap with  $\Gamma$ -convex values on a bounded hyperconvex space  $H$ . Then  $M$  has a fixed point.*

Further, Sine [S2, Corollary 3] showed that the fixed point set is hyperconvex if the values of  $M$  are nonempty closed ball intersections (which are  $\Gamma$ -convex). From this, results on common fixed points for commuting families of nonexpansive maps follow.

Since every singleton is  $\Gamma$ -convex in a hyperconvex space, the first part of Theorem 3.1 follows from Corollary 3.5.

Moreover, the following due to Sine [S2, Theorem 15] gives another example of hyperconvex spaces:

**Theorem 3.6.** *The collection  $BI(H)$  of all nonempty closed ball intersections in a hyperconvex space  $H$  is hyperconvex under the Hausdorff metric.*

This is also proved by Kirk [Ki, Lemma 2] for the case  $H$  itself is bounded.

#### 4. KAKUTANI TYPE MULTIMAPS

For a topological space  $X$  and a convex space  $Y$ , a multimap  $T : X \multimap Y$  is called a *Kakutani map* if it is upper semicontinuous (u.s.c.) and has nonempty closed convex values.

Recently, the first author obtained the following fixed point theorem [P8, Theorem 2] for Kakutani type multimaps:

**Theorem 4.1.** *Let  $(X, D; \Gamma)$  be an LG-space and  $T : X \multimap X$  a compact u.s.c. multimap with nonempty closed  $\Gamma$ -convex values. Then  $T$  has a fixed point.*

Note that, for a convex subset  $X = D$  of a locally convex t.v.s., Theorem 4.1 reduces to the well-known Himmelberg fixed point theorem.

In this section, we first show that several known Schauder type fixed point theorems on hyperconvex spaces are consequences of the following corollary of Theorem 4.1, which improves Horvath [H4, Corollary 4.4]:

**Corollary 4.2.** *Let  $(X, D; \Gamma)$  be an LG-space such that singletons are  $\Gamma$ -convex. Then any compact continuous function  $f : X \rightarrow X$  has a fixed point.*

We list known particular forms of Corollary 4.2 for hyperconvex spaces :

- (i) Espinola-Garcia [E, Lemma 3]: Let  $X$  be a compact hyperconvex space and  $f : X \rightarrow X$  a continuous map. Then  $f$  has a fixed point.
- (ii) Khamsi [K, Theorem 6]: Let  $X \in BI(H)$  be a compact subset of a hyperconvex space  $H$  and  $f : X \rightarrow X$  a continuous map. Then  $f$  has a fixed point.
- (iii) Park [P5, Theorem 7]: Let  $H$  be a hyperconvex space and  $f : H \rightarrow H$  a continuous function. If  $f$  is compact, then  $f$  has a fixed point.

Note that in view of Theorem 2.2, (i)-(iii) are all simple consequences of Corollary 4.2.

From Theorem 2.3 and Corollary 4.2, we obtain the following new result :

**Theorem 4.3.** *Let  $(X, d; \Gamma)$  be a complete LC-metric space such that  $\Gamma_{\{x\}} = \{x\}$ ,  $Z \subset X$  with  $\dim_X Z \leq 0$ , and  $\Phi : X \multimap X$  a l.s.c. multimap with nonempty closed values such that  $\Phi(x)$  is  $\Gamma$ -convex for  $x \notin Z$ . If  $\Phi$  is compact, then  $\Phi$  has a fixed point.*

*Proof.* By Theorem 2.3 with  $X = Y$ ,  $\Phi$  admits a continuous selection  $f : X \rightarrow X$ . Since  $\Phi$  is compact, so is  $f$ . Now by Corollary 4.2,  $f$  has a fixed point. This completes our proof.

From Theorems 4.1 and 4.3, we have the following:

**Theorem 4.4.** *Let  $(X, d; \Gamma)$  be a complete LC-metric space such that  $\Gamma_{\{x\}} = \{x\}$ , and  $\Phi : X \multimap X$  an u.s.c. or a l.s.c. multimap with nonempty closed  $\Gamma$ -convex values. If  $\Phi$  is compact, then  $\Phi$  has a fixed point.*

For a hyperconvex space, Theorem 4.4 reduces to the following:

**Corollary 4.5.** *Let  $H$  be a hyperconvex space and  $\Phi : H \multimap H$  a compact map with nonempty closed  $\Gamma$ -convex values. If  $\Phi$  is u.s.c. or l.s.c., then  $\Phi$  has a fixed point.*

Note that Corollary 4.5 generalizes all of (i)-(iii) to compact multimaps.

The u.s.c. case of Corollary 4.5 was given by a different method in Park [P11, Theorem 7] together with three corollaries. One of them answers the question raised by Kirk and Shin [KS, Corollary 3.5] affirmatively.

Finally, in this section, we note that it is routine to obtain fixed point results on condensing maps from corresponding results on compact maps.

## 5. FAN–BROWDER TYPE THEOREMS

From Theorem 4.3 we have the following:

**Theorem 5.1.** *Let  $(X, d; \Gamma)$  be a complete LC-metric space such that  $\Gamma_{\{x\}} = \{x\}$ ,  $Z \subset X$  with  $\dim_X Z \leq 0$ , and  $T : X \multimap X$  a multimap with nonempty closed values such that*

- (1) *for each  $x \notin Z$ ,  $T(x)$  is  $\Gamma$ -convex; and*
- (2) *for each  $y \in X$ ,  $T^-(y)$  is open in  $X$ .*

*If  $T$  is compact, then  $T$  has a fixed point.*

*Proof.* Since  $T$  has open fibers by (2),  $T$  is l.s.c. Hence, the conclusion follows from Theorem 4.3.

In order to obtain a variant of Theorem 5.1, we need the following improved version of Horvath [H4, Theorem 3.2] due to the author [P7, Theorem 8]:

**Theorem 5.2.** *Let  $X$  be a paracompact space,  $(Y, \Gamma)$  a  $C$ -space, and  $S, T : X \multimap Y$  two multimaps such that*

- (1) *for each  $x \in X$ ,  $A \in \langle S(x) \rangle$  implies  $\Gamma_A \subset T(x)$ ; and*
- (2)  *$X = \bigcup \{\text{Int } S^-(y) : y \in Y\}$ .*

*Then  $T$  has a continuous selection.*

The following is due to Komiya [Ko]:

**Corollary 5.3.** *Let  $(X, \Gamma)$  be a paracompact  $C$ -space, and  $S, T : X \multimap X$  two multimaps satisfying conditions (1) and (2) in Theorem 5.2 with  $X = Y$ . If  $(X, \Gamma)$  has the fixed point property (that is, every continuous selfmap has a fixed point), then  $T$  has a fixed point.*

Further, Komiya [Ko] also noted the following:

**Proposition 5.4.** *Let  $(X, \Gamma)$  be a paracompact LG-space. If any multimap  $T : X \multimap X$  satisfying the requirements in Corollary 5.3 with  $S = T$  has a fixed point, then any Kakutani type map  $F : X \multimap X$  has a fixed point.*

From Corollary 4.2 and Theorem 5.2, we obtain the following :

**Theorem 5.5.** *Let  $H$  be a hyperconvex space and  $S, T : H \multimap H$  two multimaps satisfying conditions (1) and (2) in Theorem 5.2. If  $T$  is compact, then  $T$  has a fixed point.*

*Proof.* Since  $H$  is a complete LC-metric space, by Theorem 5.2 with  $X = Y = H$ ,  $T$  has a continuous selection  $f : H \rightarrow H$ . Since  $T$  is compact, so is  $f$ . Hence, by (iii),  $f$  has a fixed point. This completes our proof.

From Theorem 5.1 with  $Z = \emptyset$  or Theorem 5.5 with  $S = T$ , we have the following :

**Theorem 5.6.** *Let  $H$  be a hyperconvex space and  $T : H \multimap H$  a multimap such that*

- (1) *for each  $x \in H$ ,  $T(x)$  is nonempty and  $\Gamma$ -convex; and*
- (2) *for each  $y \in H$ ,  $T^-(y)$  is open in  $H$ .*

*If  $T$  is compact, then  $T$  has a fixed point.*

*Remarks.* 1. If  $H$  is a convex space or a convex subset of a t.v.s., then the validity of Theorem 5.6 is not known yet. This is raised by Ben-El-Mechaiekh as a problem. For particular solutions, see Park [P1,12].

2. In the KKM theory, there are lot of equivalent statements of the KKM theorem and the Fan-Browder theorem. This can be done for hyperconvex spaces by using the corresponding results in [P4,5].

In case  $H$  itself is compact, Theorem 5.6 reduces to the following Fan-Browder type theorem :

**Theorem 5.7.** *Let  $H$  be a compact hyperconvex space and  $T : H \multimap H$  a multimap with values in  $BI(H)$  and open fibers. Then  $T$  has a fixed point.*

Theorem 5.7 has a generalization for  $G$ -convex spaces due to Park and Kim [PK1,2], which includes a lot of particular cases. One of them, due to Horvath [H1, Théorème 2], [H2, Theorem I.4], Park and Jeong [PJ, Theorem 4], is as follows :

**Theorem 5.8.** *Let  $X$  be a compact contractible space and  $G : X \multimap X$  a multimap satisfying*

- (a)  *$G(x)$  is open for each  $x \in X$  and  $G^-(y)$  is nonempty for each  $y \in Y$ ; and*
- (b) *for each open set  $O$  in  $X$ , the set  $\bigcap_{y \in O} G^-(y)$  is empty or contractible.*

*Then  $G$  has a fixed point.*

*Proof of Theorem 5.7 using Theorem 5.8.* Note that by Theorem 2.1, every hyperconvex space and hence every element of  $BI(H)$  is contractible. Therefore, by putting  $X = H$  and  $G = T^-$ , conditions (a) and (b) are satisfied. In fact,  $G(x) = T^-(x)$  is open and  $G^-(y) = T(y)$  belongs to  $BI(H)$ . Note that any intersection of a family in  $BI(H)$  is empty or belongs to  $BI(H)$ , and hence it is empty or contractible. Therefore by Theorem 5.8, there exists an  $x_0 \in G(x_0) = T^-(x_0)$  such that  $x_0 \in T(x_0)$ . This completes our proof.

*Remarks.* 1. If  $X$  is a convex space in Theorem 5.8 and if  $G^-(y)$  is convex for each  $y \in X$ , then we obtain the well-known Fan-Browder theorem, which has numerous applications.

2. Note that Theorem 5.7 is another application of Theorem 5.8. This would satisfy a question raised by an inconsiderate reviewer of [PJ] in MR 96m:47109.

## 6. CARISTI–KIRK–BROWDER TYPE THEOREMS

Recently, the first author [P4] generalized the well-known Caristi-Kirk-Browder fixed point theorem [C] as follows:

Let  $(X, d)$  be a quasi-metric space, where  $d$  is not necessarily symmetric. A function  $\omega : X \times X \rightarrow [0, \infty)$  is called a *W-distance* on  $X$  if the following are satisfied:

- (1)  $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$  for any  $x, y, z \in X$ ;
- (2) for any  $x \in X$ ,  $\omega(x, \cdot) : X \rightarrow [0, \infty)$  is lower semicontinuous; and
- (3) for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $\omega(z, x) \leq \delta$  and  $\omega(z, y) \leq \delta$  imply  $d(x, y) \leq \varepsilon$ .

Let  $\phi : X \times X \rightarrow (-\infty, \infty]$  be a function such that

- (4)  $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$  for any  $x, y, z \in X$ ; and
- (5)  $\phi(x, \cdot)$  is lower semicontinuous for all  $x \in X$ ;
- (6) there exists an  $x_0 \in X$  such that  $\inf_{y \in X} \phi(x_0, y) > -\infty$ .

It is known that for a quasi-metric space  $(X, d)$ , the concepts of Cauchy sequences, completeness, and Banach contractions can be defined.

The following is a particular form of Park [P6, Theorem (iii)] :

**Theorem 6.1.** *Let  $(X, d)$  be a complete quasi-metric space,  $\omega : X \times X \rightarrow [0, \infty)$  a W-distance on  $X$ , and  $\phi : X \times X \rightarrow (-\infty, \infty]$  a function satisfying (4)-(6). If a function  $f : X \rightarrow X$  satisfies*

$$\phi(x, f(x)) + \omega(x, f(x)) \leq 0 \quad \text{for all } x \in X,$$

*then  $f$  has a fixed point.*

This has several equivalent formulations and some variants. Especially, for  $\omega = d$  and  $\phi(x, y) = g(y) - g(x)$ , where  $g : X \rightarrow (-\infty, \infty]$  is a proper lower semicontinuous function bounded from below, Theorem 6.1 reduces to the Caristi-Kirk-Browder fixed point theorem, which includes the Banach contraction principle. Moreover, it is known that Theorem 6.1 and its equivalent forms have numerous applications; see [P6] and references therein.

We close this paper by noting that Theorem 6.1 is applicable to hyperconvex spaces since they are complete metric spaces, and hence we can obtain results for such spaces.

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