NEW SUBCLASSES OF GENERALIZED CONVEX SPACES

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Abstract. We give some new examples of G-convex spaces and, simultaneously, show that some abstract convexities of other authors are simple particular examples of our G-convexity.

Dedicated to Dr. Hyun-Chun Shin who founded Gyeongsang National University as the fisrt President.

0. G-convex spaces

The concept of convex sets in a topological vector space was extended to convex spaces by Lassonde [L], and further to C-spaces by Horvath [H1-3]. A number of other authors also extended the concept of convexity for various purposes. Recently, the present author unified those concepts and introduced generalized convex spaces or G-convex spaces. The study on G-convex spaces was done in a sequence of papers of Park *et al.* [PK1-6, P1-4, LP]; especially, the foundations of the KKM theory with respect to the admissible class of multimaps were established by Park and Kim [PK3].

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X, and a multimap $\Gamma : \langle D \rangle \multimap X$ such

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that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Note that $\phi_A|_{\Delta_J}$ can be regarded as ϕ_J.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X, \Gamma) = (X, X; \Gamma)$. A subset C of X is said to be Γ-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

At first, a G-convex space was defined under the additional assumption that

(∗) for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$;

see [PK1-4]. Later, it is known that this restriction is superfluous, and recent works of the author [P2-4, LP] adopt the above definition.

Major examples of other G-convex spaces than convex spaces or C-spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joo's pseudoconvex spaces, topological semilattices with path-connected intervals, and so on. For the literature, see [PK1-3, P1]. Recently, Ben-El-Mechaiekh *et al.* [BC] called (X, Γ) an *L-space* and $\Gamma : \langle X \rangle \multimap X$ an *L-structure* on X, and gave examples of L-spaces as follows: B'-simplicial convecity, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

Moreover, a number of authors investigated another abstract convexities particular to G-convex spaces for various purposes. All of those authors considered the case $X = D$, contrary to the classical works of Knaster-Kuratowski-Mazurkiewicz [KKM] and Fan [F] for the case $X \neq D$. This fact should be recognized by all peoples working in generalized abstract convexities.

In this paper, we give some new subclasses of the class of G-convex spaces and, simultaneously, show that some recent abstract convexities of other authors are simple particular examples of our G-convexity. Such subclasses are L-spaces of Ben-El-Mechaiekh *et al.* [BC], continuous images of C-spaces, Verma's generalized H-spaces [V1-4], Kulpa's simplicial structures [K], $P_{1,1}$-spaces of Forgo and Joó [FJ], generalized H-spaces of Stachó [S], and Llinares' *mc*-spaces [LL1,2].

1. Continuous images of C-spaces

If $X = D$ and Γ_A is assumed to be contractible or, more generally, ω-connected (that is, n-connected for all $n \geq 0$), and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then (X, Γ) becomes a *C-space* (or an *H-space*) due to Horvath [H1-3].

However, this concept can be extended to the one for $X \neq D$; see [PK1,2].

Now we give a new class of G-convex spaces as follows:

Proposition. *Any continuous images of C-spaces are G-convex spaces.*

Proof. Let $(X, D; \Gamma)$ be a C-space; that is, Γ_A is ω-connected for each $A \in \langle D \rangle$. Then there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A$ with $|A| = n+1$ satisfying the requirement of the definition of G-convex spaces; see Horvath [H1] and Kim [K]. Let Y be a topological space with a continuous surjection $f : X \to Y$. Let $D' := f(D)$ and for any $y \in D'$ choose an element $x_y \in f^{-1}(y) \cap D$ by the axiom of choice.

Define $\Gamma' : \langle D' \rangle \multimap Y$ by

$$\Gamma'(A') = (f \circ \Gamma)(\{x_y \in D : y \in A'\}) \text{ for any } A' \in \langle D' \rangle.$$

Then $(Y, D'; \Gamma')$ is a G-convex space. In fact, for each $A' \in \langle D' \rangle$ with $|A'| = n+1$, there exist a set $A = \{x_y \in D : y \in A'\} \in \langle D \rangle$ with $|A| = n+1$ and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Then we have $\psi_{A'} := f \circ \phi_A : \Delta_n \to (f \circ \Gamma)(A) = \Gamma'(A')$ such that $J' \in \langle A' \rangle$ implies $\psi_{A'}(\Delta_J) = (f \circ \phi_A)(\Delta_J) \subset f(\Gamma_J) = \Gamma'(J')$ where $J := \{x_y \in D : y \in J'\}$ and $\Delta_J = \Delta_{J'}$. This completes our proof.

This answers to the question raised by George Yuan at the NACA '98, Niigata, Japan, whether there are non-trivial examples of G-convex spaces which are not C-spaces.

Note that $\Gamma'(A')$ is not necessarily ω-connected since ω-connectedness is not a continuous invariant.

2. Verma's generalized H-spaces

In a sequence of recent announcements [V1-4], Verma introduced the following concept:

For a topological space X, a triple (X, H, f) is said to be a *generalized H-space* if there is a map $H : \langle X \rangle \to \mathcal{P}(X) \backslash \{\emptyset\}$ such that the following assumptions hold:

(i) For each F, G in $\langle X \rangle$, there exists a $F_1 \subset F$ such that $F_1 \subset G$ implies $H(F_1) \subset H(G)$.

(ii) For $F = \{x_1, \ldots, x_n\}$ in $\langle X \rangle$, there exists a continuous map $f : \Delta_n \to H(F)$ such that for each $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, we have
$$f(e_{i_1}, \ldots, e_{i_k}) \subset H(\{x_{i_1}, \ldots, x_{i_k}\}),$$
where elements $x_1, \ldots, x_n$ are not necessarily distinct.

Note that, in the above definition, $\mathcal{P}(X)$ is the power set of X and Δ_n denotes $(n-1)$-simplex $(e_1, \ldots, e_n)$ and $(e_{i_1}, \ldots, e_{i_k}) = \mathrm{co}\{e_{i_1}, \ldots, e_{i_k}\}$.

Apparently, this concept is motivated by that of G-convex spaces due to the author. However, this concept has at least two defects.

First, f in the above definition should be a family $\{f_A : A \in \langle X \rangle\}$ as in the definition of G-convex spaces.

Second, condition (i) is not clear. But (i) implies the following:

(i)$'$ for each $A, B \in \langle X \rangle$, $A \subset B$ implies $H(A) \subset H(B)$.

In fact, by putting $F = F_1 = A$ and $G = B$, (i) clearly implies (i)$'$.

Moreover, if $|F| = m$ and $m < n$ in (ii), then for the standard inclusion $i : \Delta_m \hookrightarrow \Delta_n$, the map $\phi_F = f \circ i : \Delta_m \to H(F)$ satisfies $\phi_F(\Delta_J) \subset H(J)$ for each $J \in \langle F \rangle$.

Therefore, a generalized H-space (X, H, f_A) with $A \in \langle X \rangle$ in the sense of Verma is simply a G-convex space (X, Γ) with $\Gamma = H$.

In view of this fact, it is quite obvious that most of results in [V1-4] are simple consequences of corresponding ones in [PK3]. For example, Verma defined a generalized H-KKM map same as our generalized KKM map [P2] and obtained a KKM type theorem for a generalized H-KKM map [V2, Theorem 2.1]. Note that this result is a particular form of [PK3, Theorem 2].

Moreover, in [V5], Verma introduced another inadequate concept of K-convex spaces.

3. Kulpa's simplicial structures

In [Ku], Kulpa introduced the following concepts:

For a topological space $X = (X, \mathcal{J})$, a continuous map $\sigma : [p_0, \ldots, p_n] \to X$ from a geometric simplex into X is called *a singular simplex* in X. We use the following notations:

$$\operatorname{dom} \sigma := [p_0, \ldots, p_n], \ \operatorname{im} \sigma := \sigma[p_0, \ldots, p_n], \ \operatorname{vert} \sigma := \{\sigma(p_0), \ldots, \sigma(p_n)\}.$$

Let Σ be the family of all singular simplices in X. A family $\mathcal{F} \subset \Sigma$ is called *a simplicial structure* in X if for each sequence of indices $0 \leq i_0 < \cdots < i_k \leq n$, we have $\sigma|_{[p_{i_0}, \ldots, p_{i_k}]} \in \mathcal{F}$.

A triple $(X, \mathcal{J}, \mathcal{F})$ is called *a topological simplicial space* and it is said to be *convex* if for each $A \in \langle X \rangle$ there exists a singular simplex $\sigma \in \mathcal{F}$ such that $A = \operatorname{vert} \sigma$.

Proposition. *A convex topological simplicial space $(X, \mathcal{J}, \mathcal{F})$ is a G-convex space (X, Γ).*

Proof. A map $\Gamma : \langle X \rangle \multimap X$ is defined by $\Gamma_A = \operatorname{im} \sigma$ whenever $A \in \langle X \rangle$, $|A| = n + 1$, and $A = \operatorname{vert} \sigma$. Let $\phi_A = \sigma : [p_0, \ldots, p_n] \to \Gamma_A$ be the singular simplex. Then for any $J = \{a_{i_0}, \ldots, a_{i_k}\} \subset \{a_0, \ldots, a_n\} = A$, we have

$$\phi_A(\Delta_J) = \phi_A\left([p_{i_0}, \ldots, p_{i_k}]\right) = \operatorname{im} \sigma_J = \Gamma_J,$$

where $\sigma_J = \sigma|_{[p_{i_0}, \ldots, p_{i_k}]}$ and $J = \operatorname{vert} \sigma_J$.

4. The $P_{1,1}$-space of Forgo and Joó

In [FJ], Forgo and Joó introduced the class of $P_{1,1}$-spaces and its subclasses. They defined the $P_{1,1}$-spaces as follows:

For a topological space X, a triple $(X, h, \mathcal{F}_{1,1})$ is called a $P_{1,1}$-*space* if the following holds:

$h : 2^X \to 2^X$ satisfies

(H1) $h(\emptyset) = \emptyset$;

(H2) $h(A) \neq \emptyset$ if $\emptyset \neq A \subset X$; and

(H3) $h(F)$ is compact for each $F \in \langle X \rangle$ and $h(A) = \bigcup\{h(F) : F \in \langle A \rangle\}$.

$\mathcal{F}_{1,1}$ is a family of continuous functions $\Phi_F : \Delta_n = [e_1, \ldots, e_n] \to X$ indexed by $F := \{x_1, \ldots, x_n\} \in \langle X \rangle$ with $|F| = n$ such that for each subsimplex $[e_{i_1}, \ldots, e_{i_k}] \subset \Delta_n$, we have

$$\Phi_F([e_{i_1}, \ldots, e_{i_k}]) \subset h(\{x_{i_1}, \ldots, x_{i_k}\}).$$

Proposition. *Any $P_{1,1}$-space is a G-convex space (X, Γ) with $\Gamma_F = h(F)$ for $F \in \langle X \rangle$.*

5. The S-convexity of Stachó

In a recent talk, Stachó [S] proposed the following concept of s-convexity:

Given any set X and a function $s : 2^X \to 2^X$, a subset $C \subset X$ is said to be s-*convex* if $s(Z) \subset C$ for any $Z \subset C$. A convexity s is *finitely generated* if $s(Z) = \emptyset$ for any infinite $Z \subset X$. A real function $\phi : X \to \mathbf{R}$ is s-*quasiconvex* if $\{x \in X : \phi(x) < \alpha\}$ and $\{x \in X : \phi(x) \leq \alpha\}$ is s-convex for any $\alpha \in \mathbf{R}$; and s-*quasiconcave* if $-\phi$ is s-quasiconvex.

Stachó is primarily interested in two extreme types of s-convesity: the interval spaces and the generalized H-convexity. His generalized H-space is almost same to the $P_{1,1}$-space of Forgo and Joó; see Section 4. Therefore, it is a particular form of G-convex spaces and we will not repeat here.

Moreover, his s-convexity can be extended as follows:

Given any set X, a nonempty subset D of X, and a function $s : \langle D \rangle \multimap X$, a subset $C \subset X$ is said to be *s-convex* if $s(Z) \subset C$ for any $Z \in \langle C \cap D \rangle$.

With this definition, it would be possible to generalize some of his new results in [S].

6. Llinares' MC-spaces

In [LL1,2], Llinares introduced an abstract convexity defined by mc-spaces as follows:

A topological space is an *mc-space* or has an *mc-structure* if for any $A = \{a_0, a_1, \ldots, a_n\} \in \langle X \rangle$, there exist a subset $\{b_0, b_1, \ldots, b_n\}$ of X and a family of functions

$$P_i^A : X \times [0,1] \to X, \quad i = 0, 1, \ldots, n$$

such that

(1) $P_i^A(x, 0) = x$ and $P_i^A(x, 1) = b_i$ for all $x \in X$; and

(2) the function

$$G_A : [0,1]^n \to X$$

given by

$$G_A(t_0, t_1, \ldots, t_{n-1}) = P_0^A(\cdots(P_{n-1}^A(P_n^A(a_n, 1), t_{n-1}), t_{n-2}), \ldots), t_0)$$

is continuous.

From an mc-structure Llinares [L2] defined an abstract convexity given by the family of those sets which are stable under the function G_A. To define his convexity he needs the following:

For a subset Z of an mc-space X and for any $A \in \langle X \rangle$ such that $A \cap Z \neq \emptyset$, $A \cap Z = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\}$, $i_0 < i_1 < \cdots < i_m$, the *restriction of the function* G_A *to* Z is defined by as follows:

$$G_{A|Z} : [0,1]^m \to X,$$
$$G_{A|Z}(t) = P^A_{i_0}(\cdots(P^A_{i_{m-1}}(P^A_{i_m}(a_{i_m}, 1), t_{i_{m-1}}), \cdots), t_{i_0}),$$

where $P^A_{i_k}$ are the functions associated with the elements $a_{i_k} \in A \cap Z$.

If $G_{A|Z}([0,1]^m) \subset Z$, then Z is called an *mc-set* of X. Note that the family of mc-sets is stable under arbitrary intersections.

Proposition. *An mc-space X with the above convexity is a G-convex space (X, Γ) with $\Gamma : \langle X \rangle \multimap X$ given by*

$$\Gamma_A := G_A([0,1]^n) \text{ for each } A \in \langle X \rangle,\ |A| = n + 1;$$
$$\Gamma_J := G_{A|J}([0,1]^m) \text{ for each } J \in \langle A \rangle,\ |J| = m + 1;$$

and

$$\phi_A|_{\Delta_J} = \phi_J.$$

Proof. For each $n \geq 0$, let $g_n : \Delta_n \to [0,1]^n$ be a continuous function given by

$$g_n : u = \sum_{i=0}^{n} \lambda_i(u) e_i \mapsto (\lambda_0(u), \ldots, \lambda_{n-1}(u))$$

for $u \in \Delta_n$, and

$$\phi_J := G_{A|Z} \circ g_m : \Delta_m \to \Gamma_J$$

for each $A \in \langle X \rangle$ and $J \in \langle A \rangle$ with $|J| = m + 1$. Then

$$\phi_A(\Delta_J) = \phi_J(\Delta_J) = (G_{A|Z} \circ g_m)(\Delta_J) = G_{A|Z}([0,1]^m) = \Gamma_J.$$

This completes our proof.

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