Journal of Computational and Applied Mathematics 113 (2000) 83–91

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

Acyclic versions of the von Neumann and Nash equilibrium theorems

Sehie Park *

Department of Mathematics, Seoul National University, Seoul 151-742, South Korea

Received 9 September 1998

Abstract

Applying a fixed point theorem for compact compositions of acyclic maps, we obtain acyclic versions of the von Neumann intersection theorem, the minimax theorem, the Nash equilibrium theorem, and others. © 2000 Elsevier Science B.V. All rights reserved.

MSC: primary 90A14; 47H10; secondary 54C60; 55M20

Keywords: Multimap (map); Closed map; Compact map; Upper semicontinuous (u.s.c.); Lower semicontinuous (l.s.c.); Kakutani map; Acyclic map; Berge's theorem; von Neumann intersection theorem; Quasi-variational inequality; Nash equilibrium

0. Introduction

In 1991, the author obtained a Schauder-type fixed point theorem for compact acyclic multimaps [14], which generalizes well-known theorems due to Himmelberg [7] and Kakutani [9] by replacing the convex values of multimaps by the acyclic values. Recently, in [17], this theorem is extended to compact acyclic maps defined on convex subsets of not-necessarily locally convex topological spaces, and applied to existence of solutions of quasi-equilibrium problems in admissible topological vector spaces (in the sense of Klee). For further generalizations of our theorem, see [20,15,16,18], where it was shown that compact acyclic maps in the aforementioned theorems can be replaced by compact compositions of acyclic maps or closed compact better admissible maps.

Applying our theorem for compact compositions of acyclic maps, in this paper, we obtain acyclic versions of the von Neumann intersection theorem, the minimax theorem, the Nash equilibrium theorem, and others.

* Fax: 00822-887-4694.

E-mail address: shpark@math.snu.ac.kr (S. Park)

PII: S0377-0427(99)00246-0

The main result (Theorem 1) is a collectively fixed point theorem. This is applied to obtain generalizations (Theorems 2 and 3) of the von Neumann intersection theorem and acyclic versions (Theorems 4 and 5) of the von Neumann minimax theorem. Moreover, we obtain a quasi-equilibrium theorem (Theorem 6), from which we deduce an acyclic version (Theorem 7) of the Nash equilibrium theorem.

1. Preliminaries

All spaces are assumed to be Hausdorff and a t.v.s. means a topological vector space.

A *multimap* or *map* $F : X \multimap Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) = \{x \in X\colon\ y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{F(x)\colon x \in A\}$. For any $B \subset Y$, the (lower) inverse of B under F is defined by

$$F^-(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}.$$

For any relation $R \subset X \times Y$, R^- denotes its inverse relation in $Y \times X$.

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *closed* if its graph

$$\mathrm{Gr}(F) = \{(x, y)\colon\ y \in F(x),\ x \in X\}$$

is closed in $X \times Y$, and *compact* if $F(X)$ is contained in a compact subset of Y.

$F : X \multimap Y$ is said to be *upper semicontinuous* (*u.s.c.*) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (*l.s.c.*) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

If F is u.s.c. with closed values, then F is closed. The converse is true whenever Y is compact.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a t.v.s., convex ⇒ star-shaped ⇒ contractible ⇒ acyclic ⇒ connected, and not conversely.

For topological spaces X and Y, a map $F : X \multimap Y$ is called a *Kakutani map* whenever Y is a convex subset of a t.v.s. and F is u.s.c. with compact convex values; and an *acyclic map* whenever F is u.s.c. with compact acyclic values.

Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X, Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

The following is a particular form of our previous work [15,16,18,20]:

Theorem A. *Let X be a nonempty convex subset of a locally convex t.v.s. E and $F \in \mathbb{V}_c(X, X)$. If F is compact, then F has a fixed point $x_0 \in X$; that is, $x_0 \in F(x_0)$.*

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well known that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are ℓ^p, $L^p(0, 1)$, H^p for $0 < p < 1$, and many others; see [16,18] and references therein.

The following particular form of a fixed point theorem due to the author [16,18] is the basis of our arguments in this paper. We give its simple proof for the sake of completeness.

Theorem B. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $F \in \mathbb{V}_c(X,X)$ has a fixed point.*

Proof. Let $\mathscr{V}$ be a fundamental system of neighborhoods of the origin 0 of E. Since F is closed and compact, it is sufficient to show that for any $V \in \mathscr{V}$, there exists an $x_V \in X$ such that $(x_V + V) \cap F(x_V) \neq \emptyset$.

Since $\overline{F(X)}$ is a compact subset of the admissible subset X, there exist a continuous map $h : \overline{F(X)} \to X$ and a finite-dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{F(X)}$ and $h(\overline{F(X)}) \subset L \cap X$. Let $M := h(\overline{F(X)})$. Then M is a compact subset of L and hence $P := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{F(X)} \to P$ and $F|_P : P \multimap \overline{F(X)}$. Since $h \circ (F|_P) \in \mathbb{V}_c(P,P)$, by Theorem A, it has a fixed point $x_V \in P$; that is, $x_V \in hF(x_V)$ and hence $x_V = h(y)$ for some $y \in F(x_V) \subset \overline{F(X)}$. Since $y - h(y) \in V$, we have $y \in h(y) + V = x_V + V$. Therefore, $(x_V + V) \cap F(x_V) \neq \emptyset$. □

Recall that a real-valued function $f : X \to \mathbb{R}$ on a topological space is *lower* [resp. *upper*] *semicontinuous* (*l.s.c.*) [resp. *u.s.c.*] if $\{x \in X\colon f(x) > r\}$ [resp. $\{x \in X\colon f(x) < r\}$ is open for each $r \in \mathbb{R}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X\colon f(x) > r\}$ [resp. $\{x \in X\colon f(x) < r\}$] is convex for each $r \in \mathbb{R}$.

We need the following [2]:

Berge's Theorem. *Let X and Y be topological spaces, $f : X \times Y \to \mathbb{R}$ a real function, $F : X \multimap Y$ a multimap, and*

$$\hat{f}(x) := \sup_{y \in F(x)} f(x,y), \qquad G(x) := \{y \in F(x)\colon f(x,y) = \hat{f}(x)\} \quad \textit{for } x \in X.$$

(a) *If f is u.s.c. and F is u.s.c. with compact values, then $\hat{f}$ is u.s.c.*

(b) *If f is l.s.c. and F is l.s.c., then $\hat{f}$ is l.s.c.*

(c) *If f is continuous and F is continuous with compact values, then $\hat{f}$ is continuous and G is u.s.c.*

Let $\{X_i\}_{i \in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j \in I} X_j \quad \text{and} \quad X^i = \prod_{j \in I \setminus \{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$, the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) = \{y_i \in X_i\colon [x^i, y_i] \in A\} \quad \text{and} \quad A(x_i) = \{y^i \in X^i\colon [y^i, x_i] \in A\}.$$

2. Intersection theorems

We begin with the following collectively fixed point theorem, which is equivalent to Theorem B for $\mathbb{V}(X,X)$:

Theorem 1. *Let $\{X_i\}_{i=1}^n$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and T_i: $X = \prod_{j=1}^n X_j \multimap K_i$ an acyclic map for each i, $1 \leqslant i \leqslant n$. If X is admissible in the t.v.s. $E = \prod_{j=1}^n E_j$, there exists an $\hat{x} \in K = \prod_{j=1}^n K_j$ such that $\hat{x}_i \in T_i(\hat{x})$ for each i.*

Proof. Define $T : X \multimap K$ by $T(x) = \prod_{j=1}^n T_j(x)$ for each $x \in X$. Then T is clearly a closed map. Since the product of two acyclic sets is acyclic by the Künneth theorem, T is an acyclic map. Since K is compact in X, by Theorem B, T has a fixed point $\hat{x} \in K$; that is, $\hat{x} \in T(\hat{x})$ and $\hat{x}_i \in T_i(\hat{x})$. □

Similarly, we can obtain a more general result:

Theorem 1′. *Let I be any index set, $\{X_i\}_{i\in I}$ a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and $T_i : X = \prod_{j\in I} X_j \multimap K_i$ an u.s.c. map for each $i \in I$. Suppose that, for each $x \in X$, $T_i(x)$ is closed convex except a finite number of i's for which $T_i(x)$ is closed acyclic. If X is admissible in $E = \prod_{j\in I} E_j$, then there exists an $\hat{x} \in K = \prod_{j\in I} K_j$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

Remark. (1) If $n = 1$, then Theorem 1 reduces to Theorem B for $\mathbb{V}(X,X)$.

(2) If each T_i is convex-valued and each E_i is locally convex, Theorem 1′ reduces to Idzik [8, Theorem 5].

From Theorem 1, we obtain the following von Neumann-type intersection theorem:

Theorem 2. *Let $\{X_i\}_{i=1}^n$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of $X = \prod_{j=1}^n X_j$ such that $A_i(x^i)$ is an acyclic subset of K_i for each $x^i \in X^i$, where $1 \leqslant i \leqslant n$. If X is admissible in $E = \prod_{j=1}^n E_j$, then $\bigcap_{j=1}^n A_j \neq \emptyset$.*

Proof. We use Theorem 1 with $T_i : X \multimap K_i$ defined by $T_i(x) = A_i(x^i)$ for $x \in X$. Then, for each $x \in X$, we have

$$\begin{aligned}(x, y) \in \mathrm{Gr}(T_i) &\Leftrightarrow (x_i, x^i) \in X_i \times X^i \quad \text{and} \quad y \in A_i(x^i) \subset K_i \\ &\Leftrightarrow (x_i, x^i, y) \in X_i \times (A_i \cap (X^i \times K_i)),\end{aligned}$$

which implies that $\mathrm{Gr}(T_i)$ is closed in $X \times K_i$. Hence, each T_i is a compact closed map with acyclic values; that is, T_i is an acyclic map. Therefore, by Theorem 1, there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for all i. Since $\hat{x}_i \in K_i \subset X_i$, we have $\hat{x} = [\hat{x}^i, \hat{x}_i] \in A_i$ for all i. □

Similarly, we can obtain a more general result than Theorem 2 as follows:

Theorem 2′. *Let I be any index set, $\{X_i\}_{i\in I}$ a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of $X=\prod_{j\in I} X_j$ for each $i\in I$. Suppose that for each $x^i\in X^i$, $A_i(x^i)$ is a convex subset of K_i except a finite number of i's for which $A_i(x^i)$ is an acyclic subset of K_i. If X is admissible in $E=\prod_{j\in I} E_j$, then $\bigcap_{j\in I} A_j\neq\emptyset$.*

Remark. (1) If $I=\{1,2\}$, E_i are Euclidean, $X_i=K_i$, and $A_i(x^i)$ are nonempty and convex, then Theorem 2 reduces to the intersection theorem of von Neumann [13].

(2) Theorem 2′ holds without the admissibility of X whenever each $A_i(x^i)$ is nonempty and convex; see [5, Theorem 2, 8, Corollary 1].

From Theorem B, we have the following intersection theorem:

Theorem 3. *Let X be a compact space and Y an admissible compact convex subset of a t.v.s. E. Let A and B be two closed subsets of $X\times Y$ such that*

(1) *for each $x\in X$, $A(x)=\{y\in Y\colon (x,y)\in A\}$ is acyclic*; *and*
(2) *for each $y\in Y$, $B(y)=\{x\in X\colon (x,y)\in B\}$ is acyclic.*

Then $A\cap B\neq\emptyset$.

Proof. Define two multimaps $S:X\multimap Y$ and $T:Y\multimap X$ by $S(x)=A(x)$ and $T(y)=B(y)$ for $x\in X$ and $y\in Y$. Then S has closed graph $\mathrm{Gr}(S)=A$ and hence, has closed acyclic values. Since Y is compact, S is u.s.c. and hence $S\in\mathbb{V}(X,Y)$. Similarly, T has closed graph $\mathrm{Gr}(T)=B^-$ in $Y\times X$ and hence, has closed acyclic values. Since X is compact, T is u.s.c. and $T\in\mathbb{V}(Y,X)$. Note that $S\circ T\in\mathbb{V}_c(Y,Y)$ is compact. Therefore, by Theorem B, $S\circ T$ has a fixed point $y_0\in Y$; that is, $y_0\in(S\circ T)(y_0)$. Then there exists an $x_0\in T(y_0)$ such that $y_0\in S(x_0)$; that is,

$$(x_0,y_0)\in\mathrm{Gr}(S)=A \quad\text{and}\quad (x_0,y_0)\in\mathrm{Gr}(T^-)=B. \qquad \square$$

Remark. If E is locally convex and if the acyclicity is replaced by "nonempty and convex", then Theorem 3 reduces to Chang [3, Theorem 3], which generalizes the intersection theorem of von Neumann [13]. Note that Chang's method relied on some results about separating disjoint graphs.

3. Minimax theorems

From Theorem 3, we have the following von Neumann-type minimax theorem:

Theorem 4. *Let X be a compact space and Y an admissible compact convex subset of a t.v.s., and $f:X\times Y\to\mathbb{R}$ a continuous real function. Suppose that for each $x_0\in X$ and $y_0\in Y$, the sets*

$$\left\{x\in X\colon f(x,y_0)=\max_{\zeta\in X} f(\zeta,y_0)\right\}$$

and

$$\left\{y \in Y\colon\ f(x_0, y) = \min_{\eta\in Y} f(x_0, \eta)\right\}$$

are acyclic. Then we have

$$\max_{x\in X}\min_{y\in Y} f(x, y) = \min_{y\in Y}\max_{x\in X} f(x, y).$$

Proof. Let

$$S(x) := \left\{y \in Y\colon\ f(x, y) = \min_{\eta\in Y} f(x, \eta)\right\}$$

and

$$T(y) := \left\{x \in X\colon\ f(x, y) = \max_{\zeta\in X} f(\zeta, y)\right\}.$$

Since f is continuous and X, Y are compact, each $S(x)$ and $T(y)$ are nonempty and closed for all $x \in X$ and $y \in Y$. Moreover, by Berge's theorem, $S : X \multimap Y$ and $T : Y \multimap X$ are u.s.c. with closed values. Therefore $A := \mathrm{Gr}(S)$ and $B := \mathrm{Gr}(T^-)$ are closed subsets in $X \times Y$. Therefore, by Theorem 3, there exists an $(x_0, y_0) \in A \cap B$; that is,

$$\max_{x\in X} f(x, y_0) = f(x_0, y_0) = \min_{y\in Y} f(x_0, y).$$

This implies

$$\begin{aligned}\min_{y\in Y}\max_{x\in X} f(x, y) &\leqslant \max_{x\in X} f(x, y_0) = f(x_0, y_0)\\ &= \min_{y\in Y} f(x_0, y) \leqslant \max_{x\in X}\min_{y\in Y} f(x, y).\end{aligned}$$

On the other hand, we clearly have

$$\min_{y\in Y}\max_{x\in X} f(x, y) \geqslant \max_{x\in X}\min_{y\in Y} f(x, y).$$

Therefore, we have the conclusion. □

Remark. (1) In the proof, we notice the existence of a saddle point $(x_0, y_0) \in X \times Y$ under the hypothesis of Theorem 4; that is,

$$\max_{x\in X} f(x, y_0) = f(x_0, y_0) = \min_{y\in Y} f(x_0, y).$$

(2) For Euclidean spaces or locally convex t.v.s., if acyclicity is replaced by convexity, then Theorem 4 reduces to the von Neumann minimax theorem [12] or Fan [5, Theorem 3], respectively.

(3) In case the acyclicity is replaced by convexity, Chang [3, Theorem 4] obtained Theorem 4 under the assumption that X is a compact subset of any t.v.s. and Y is a compact convex subset of a locally convex t.v.s.

(4) Note that Debreu [4] obtained Theorem 4 for the case X and Y are contractible polyhedra and acyclicity is replaced by contractibility. Since Theorem A for $\mathbb{V}(X, X)$ holds for a contractible polyhedra X by the classical results of Eilenberg-Montgomery or Begle [1], by following our method, we can obtain Debreu's result; see [19].

The following generalization of the von Neumann minimax theorem [12] is a simple consequence of Theorems 3 and 4.

Theorem 5. *Let X, Y, and f be the same as in Theorem 4. Suppose that*

(1) *for every $x \in X$ and $\alpha \in \mathbb{R}$, $\{y \in Y\colon f(x,y) \leqslant \alpha\}$ is acyclic; and*
(2) *for every $y \in Y$ and $\beta \in \mathbb{R}$, $\{x \in X\colon f(x,y) \geqslant \beta\}$ is acyclic.*

Then we have

$$\max_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \max_{x\in X} f(x,y).$$

4. Equilibrium theorems

Theorem 1 has the following equivalent form of quasi-equilibrium theorem:

Theorem 6. *Let $\{X_i\}_{i=1}^n$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, $S_i : X \multimap K_i$ a closed map, and $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ u.s.c. functions for each i. Suppose that for each i,*

(i) *$g_i(x) \leqslant f_i(x)$ for each $x \in X$;*
(ii) *the function M_i defined on X by*

$$M_i(x) = \max_{y\in S_i(x)} g_i(x^i, y) \quad \text{for } x \in X$$

is l.s.c.; and
(iii) *for each $x \in X$, the set*

$$\{y \in S_i(x)\colon f_i(x^i, y) \geqslant M_i(x)\}$$

is acyclic.

If X is admissible in $E = \prod_{j=1}^n E_j$, then there exists an $\hat{x} \in K$ such that for each i,

$$\hat{x}_i \in S_i(\hat{x}) \quad \text{and} \quad f_i(\hat{x}^i, \hat{x}_i) \geqslant g_i(\hat{x}^i, y) \quad \text{for all} \quad y \in S_i(\hat{x}).$$

Proof. For each i, $1 \leqslant i \leqslant n$, define a map $T_i : X \multimap K_i$ by

$$T_i(x) = \{y \in S_i(x)\colon f_i(x^i, y) \geqslant M_i(x)\}$$

for $x \in X$. Note that each $T_i(x)$ is nonempty by (ii) since $S_i(x)$ is compact and $g_i(x^i, \cdot)$ is u.s.c. on $S_i(x)$. We show that $\mathrm{Gr}(T_i)$ is closed in $X \times K_i$. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f_i(x^i, y) &\geqslant \overline{\lim_\alpha}\, f_i(x^i_\alpha, y_\alpha) \geqslant \overline{\lim_\alpha}\, M_i(x_\alpha) \\ &\geqslant \underline{\lim_\alpha}\, M_i(x_\alpha) \geqslant M_i(x) \end{aligned}$$

and, since $\mathrm{Gr}(S_i)$ is closed in $X \times K_i$, $y_\alpha \in S_i(x_\alpha)$ implies $y \in S_i(x)$. Hence, $(x, y) \in \mathrm{Gr}(T_i)$. Since T_i is compact, each T_i is acyclic by (iii). Therefore, by Theorem 1, there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$. □

Remark. (1) If $f_i \equiv g_i \equiv 0$ for all i, then Theorem 6 reduces to Theorem 1.

(2) If the set in (iii) is nonempty convex, $f_i \equiv g_i$ is continuous, and S_i is continuous, then Theorem 6 can be extended to any family $\{X_i\}_{i\in I}$. In case each E_i is locally convex, this is due to Idzik [8, Theorem 7].

From Theorem 6, we have the following generalization of the Nash equilibrium theorem:

Theorem 7. *Let $\{X_i\}_{i=1}^n$ be a family of nonempty compact convex subsets, each in a t.v.s. E_i and for each i, let $f_i : X \to \mathbb{R}$ be a continuous function such that*
(1) *for each $x^i \in X^i$ and each $\alpha \in \mathbb{R}$, the set*

$$\{x_i \in X_i\colon\ f_i(x^i, x_i) \geqslant \alpha\}$$

is empty or acyclic.

If X is admissible in $E = \prod_{j=1}^n E_j$, there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i(\hat{x}^i, y_i) \quad \text{for all } i,\ 1 \leqslant i \leqslant n.$$

Proof. Apply Theorem 6 with $f_i = g_i$ and $S_i(x) = X_i$ for $x \in X$. Then (ii) follows from Berge's theorem, and the set in (iii) is nonempty and acyclic by (1). Therefore, we have the conclusion. □

Remark. (1) If $x_i \mapsto f_i(x^i, x_i)$ is quasiconcave for each $x^i \in X^i$ and E_i is Euclidean for each $i \in I$, then Theorem 7 reduces to Nash [11, Theorem].

(2) If the acyclicity is replaced by convexity, then Theorem 7 holds without assuming the admissibility of X; see [6, Theorem 4]. This was extended for any infinite family $\{X_i\}_{i\in I}$ by Ma [10, Theorem 4].

Acknowledgements

This research is supported in part by the Non-Directed Research Fund, Korea Research Foundation, 1997. This was presented as an invited talk at the International Conference, Banaras Hindu University, Varanasi, India, 2–5 March 1998, and its abstract appeared in Progress of Math. 32 (1998) 4–5.

References

[1] E.G. Begle, A fixed point theorem, Ann. Math. 51 (1950) 544–550.
[2] C. Berge, Espaces Topologique, Dunod, Paris, 1959.
[3] S.-Y. Chang, Separation and von Neumann intersection theorems, Proc. Amer. Math. Soc. 112 (1991) 1149–1152.
[4] G. Debreu, A social equilibrium existence theorem, Proc. Natl. Acad. Sci. USA 38 (1952) 886–893.
[5] Ky Fan, Fixed point and minimax theorems in locally convex linear spaces, Proc. Natl. Acad. Sci. USA 38 (1952) 121–126.
[6] Ky Fan, Applications of a theorem concerning sets with convex sections, Math. Ann. 163 (1966) 189–203.
[7] C.J. Himmelberg, Fixed points of compact multifunctions, J. Math. Anal. Appl. 38 (1972) 205–207.
[8] A. Idzik, Remarks on Himmelberg's fixed point theorem, Bull. Acad. Polon. Sci. 26 (1978) 909–912.
[9] S. Kakutani, A generalization of Brouwer's fixed-point theorem, Duke Math. J. 8 (1941) 457–459.

[10] T.-W. Ma, On sets with convex sections, J. Math. Anal. Appl. 27 (1969) 413–416.
[11] J. Nash, Non-cooperative games, Ann. Math. 54 (1951) 286–293.
[12] J. von Neumann, Zur Theorie der Gesellschaftsspiele, Math. Ann. 100 (1928) 295–320.
[13] J. von Neumann, Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, Ergeb. eines Math. Kolloq. 8 (1937) 73–83, re-printed in: Rev. Economic Studies XIII (33) (1945–46), pp. 1–9.
[14] S. Park, Some coincidence theorems on acyclic multifunctions and applications to KKM theory, in: K.-K. Tan (Ed.), Fixed Point Theory and Applications, World Sci., River Edge, NJ, 1992, pp. 248–277.
[15] S. Park, Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps, J. Korean Math. Soc. 31 (1994) 493–519.
[16] S. Park, Fixed points of the better admissible multimaps, Math. Sci. Res. Hot-Line 1 (9) (1997) 1–6.
[17] S. Park, Fixed points and quasi-equilibrium problems, Math. Comp. Modelling, to appear.
[18] S. Park, A unified fixed point theory of multimaps on topological vector spaces, J. Korean Math. Soc. 35 (1998) 803–829.
[19] S. Park, Remarks on a social equilibrium existence theorem of G. Debreu, Appl. Math. Lett. 11(5) (1998) 51–54.
[20] S. Park, S.P. Singh, B. Watson, Some fixed point theorems for composites of acyclic maps, Proc. Amer. Math. Soc. 121 (1994) 1151–1158.