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Remarks on Saddle Points in Nonconvex Sets

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Abstract—Smol'yakov's saddle point theorem is generalized to admissible sets (in the sense of Klee). Moreover, convexity of the involved sets in the theorem can be replaced by acyclicity, and continuity of the involved functions by lower or upper semicontinuity.

Keywords—Two-person zero-sum game, Compact closed multimap, Admissible set (in the sense of Klee), Acyclic set, Dependent saddle point.

Recently, Smol'yakov [1] obtained a general theorem on the existence of a saddle point in two-person zero-sum games with mutually dependent sets of strategies on a nonconvex set in a topological vector space. This new theorem seems to have more general applicability than known results of the same nature. Actually, an example given in [1] can be applied by the new theorem, but not by all known theorems.

In the present paper, we generalize the main result of [1] from several points of view.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus, any nonempty convex or star-shaped subset of a topological vector space is acyclic.

We need the following well-known fact; see [2, Lemma 2.1].

LEMMA. *Let X be a Hausdorff compact space and Y a convex subset of a topological vector space E. If $f : X \to Y$ is a continuous function such that $f(X) \subset P \subset Y$, where P is a compact convex set in a finite dimensional subspace of E, and $F : Y \multimap X$ is an upper semicontinuous multimap with nonempty closed acyclic values, then the composition $f \circ F$ has a fixed point $y_0 \in P$; that is, $y_0 \in (f \circ F)(y_0)$.*

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A nonempty subset X of a topological vector space E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well known that every nonempty convex subset of a locally convex topological vector space is admissible. Other examples of admissible topological vector spaces are $\ell^p, L^p(0,1), H^p$ for $0 < p < 1$, and many others. For more details and other terminology, see [3,4] and references therein.

Our main tool is the following particular form of a fixed-point theorem recently due to the first author [3,4], and we give its proof for completeness.

THEOREM 1. *Let W be an admissible convex subset of a Hausdorff topological vector space E and $P : W \multimap W$ a compact closed multimap with closed acyclic values. Then P has a fixed point.*

PROOF. Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin 0 of E. Since P is a compact closed multimap, it is sufficient to show that for any $V \in \mathcal{V}$, there exists an $x_V \in V$ such that $(x_V + V) \cap P(x_V) \neq \emptyset$.

Let V be any set in $\mathcal{V}$. Since $\overline{P(W)}$ is a compact subset of the admissible set W, there exist a continuous map $h : \overline{P(W)} \to W$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{P(W)}$ and $h(\overline{P(W)}) \subset L \cap W$.

Let $M := h(\overline{P(W)})$. Then M is a compact subset of L and $K := \mathrm{co}M$ is a compact convex subset of $L \cap W$. Since the compact closed map P is upper semicontinuous, by the lemma, the composition $h \circ P|_K : K \to K$ has a fixed point $x_V \in K$. This means that there exists $y_V \in P(x_V)$ such that $x_V = h(y_V)$. From $y_V - h(y_V) \in V$, it follows that $y_V \in h(y_V) + V = x_V + V$. Therefore, we conclude that $(x_V + V) \cap P(x_V) \neq \emptyset$. This completes the proof.

Smol'yakov [1] considered the following game. The first player selects a strategy (point) q from some space Q in such a way that the functional $I(q, r)$ will be minimized, and the second player selects a strategy (point) r from some space R in such a way that this functional will be maximized; where (q, r) belongs to some set $D \subset Q \times R$ and $I : D \to \mathbb{R}$ is the payoff function.

Let $\mathrm{Pr}_Q\, D$ denote the projection of the set D into the space Q and $D(r)$ a section of D. We can then say that the first player can select strategies (points) in the set $\mathrm{Pr}_Q\, D(r)$ or $\mathrm{Pr}_Q\, D$, and the second player can select strategies in the set $\mathrm{Pr}_R\, D(q)$ or $\mathrm{Pr}_R\, D$. The sections $D(q)$ will be treated as subsets of the set D or the space R, depending on the context. The sections $D(r)$ will be treated similarly.

And then Smol'yakov assumed the following about the functional and the set of participant strategies.

ASSUMPTIONS. *The sets Q and R are compact convex subsets of locally convex Hausdorff topological vector spaces; D is a connected closed set in $Q \times R$ such that any of its nonempty sections $D(q)$, where $q \in \mathrm{Pr}_Q\, D$, and $D(r)$, where $r \in \mathrm{Pr}_R\, D$, are convex; the functional $I(q, r)$ is a closed bounded function; for each $r \in R, I(\cdot, r)$ is a continuous convex function from Q into $\mathbb{R}$; and for each $q \in Q, I(q, \cdot)$ is a continuous concave function from R into $\mathbb{R}$.*

Smol'yakov [1] defined the following.

DEFINITION. *A point $(q^*, r^*) \in D \subset Q \times R$ is called a dependent saddle point if*

$$I(q^*, r) \leq I(q^*, r^*) \leq I(q, r^*), \qquad \text{for all } q \in D(^*) \text{ and } r \in D(q^*).$$

Now we prove our mail result as follows under much general situation than Smol'yakov's assumptions.

THEOREM 2. *Let Q and R be convex subsets of Hausdorff topoligical vector spaces E and F, respectively, and D a connected compact set in $Q \times R$. Let $I : D \to \mathbb{R}$ be a payoff function in a two-person zero-sum game on D such that I has a closed path, $I(\cdot, r)$ is lower semicontinuous on $D(r)$ for each $r \in \mathrm{Pr}_R\, D$, and $I(q, \cdot)$ is upper semicontinuous on $D(q)$ for each $q \in \mathrm{Pr}_Q\, D$. Let $W = \mathrm{co}\{\mathrm{Pr}_Q\, D \times \mathrm{Pr}_R\, D\}$ be an admissible convex subset of $E \times F$ such that each section $W(q)$ or $W(r)$ of W contains $D(q)$ or $D(r)$, respectively. Suppose that*

$$P_1(q,r) = \{y \in D(r) : I(y,r) = \min_{q' \in D(r)} I(q',r)\}$$

and

$$P_2(q,r) = \{z \in D(q) : I(q,z) = \max_{r' \in D(q)} I(q,r')\}$$

are acyclic.

Then

(1) *the map $P = (P_1, P_2) : W \multimap W$ is a closed map such that $P(W) \subset D$;*
(2) *P has a fixed point $w \in D$; and*
(3) *w is a dependent saddle point.*

PROOF. For each $(q,r) \in W$, note that $P(q,r) = P_1(q,r) \times P_2(q,r)$ is nonempty since $I(\cdot, r)$ is lower semicontinuous on the compact set $D(r)$ and $I(q,\cdot)$ is upper semicontinuous on the compact set $D(q)$. Note that the set $P(q,r)$ is acyclic.

(a) We claim that P is a closed map. Let $(q_\alpha, r_\alpha) \in D$ be any net converging to a point $(q_0, r_0) \in D$ and let $(y_\alpha, z_\alpha) \in P(q_\alpha, r_\alpha)$ be any net such that $(y_\alpha, z_\alpha) \to (y_0, z_0)$. To show $(y_0, z_0) \in P(q_0, r_0)$, take any point $(y,z) \in P(q_0, r_0)$. Since $I(\cdot, z)$ is lower semicontinuous, we have $I(q_0, z) \le \liminf I(q_\alpha, z)$. For each α, by definition of P_2,

$$I(q_\alpha, z) \le \max_{r_\alpha \in D(q_\alpha)} I(q_\alpha, r_\alpha) = I(q_\alpha, z_\alpha),$$

where $q_\alpha \to q_0$ and $z_\alpha \to z_0$. Since I is a closed map on D,

$$I(q_0, z) \le \lim I(q_\alpha, z_\alpha) = I(q_0, z_0).$$

A similar argument shows that $I(y, r_0) \ge \lim I(y_\alpha, r_\alpha) = I(y_0, r_0)$. Therefore, it follows by definition of P_1 and P_2 that $(y_0, z_0) \in P(q_0, r_0)$.

(b) By Theorem 1, P has a fixed point $w \in D$.
(c) w is a dependent saddle point by definition of P.

Note that under the assumptions, Theorem 2 reduces to [1, Theorem 2], where

(1) E and F are locally convex;
(2) $I(\cdot, r)$ and $I(q, \cdot)$ are continuous; and
(3) $P_1(q,r)$ and $P_2(q,r)$ are convex.

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