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On generalizations of the Ekeland-type variational principles[1]

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1. Introduction

Since the celebrated variational principle of Ekeland [4] for approximate solutions of non-convex minimization problems appeared in 1972, it has received a great deal of attention and has been applied to numerous problems in various fields as shown in [5, 6]. There have also appeared many extensions or equivalent formulations of the principle as seen in the references.

In a recent work of the author and Kang [20], known extensions or equivalent formulations of Ekeland's variational principle were unified in a far-reaching general form. Since then, there have also appeared some other extensions of the principle. In fact, Oettli and Théra [13] obtained another extended equivalent form of the principle as well as some applications. In the same spirit, Blum and Oettli [1] gave an extension of Takahashi's non-convex minimization theorem [23]. Moreover, Kada et al. [8] improved Takahashi's theorem replacing the involved metric by a newly defined W-distance and gave some applications.

The aim of this paper is to unify the results in [1,8,13] along the lines of [20] and to improve the equivalent formulations of Ekeland's principle in various aspects. In fact, we obtain far-reaching generalized forms of Ekeland's principle and its six equivalents (Theorems 1, 1′, and 2). We also show that one of our formulations readily implies the principle (Theorem 3). Moreover, as a simple application, we give an extended

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form (Theorem 4) of a fixed point theorem of Downing and Kirk [3]. Finally, we add historical remarks.

Note that this paper is a refined version of our earlier announcement on this topic [19].

2. Main results

Kada et al. [8] introduced the concept of W-distances for a metric space (X,d) as follows:

A function $\omega : X\times X\to[0,\infty)$ is called a *W-distance* on X if the following are satisfied:

(1) $\omega(x,z)\le\omega(x,y)+\omega(y,z)$ for any $x,y,z\in X$;
(2) $\omega(x,\cdot):X\to[0,\infty)$ is lower semicontinuous for any $x\in X$; and
(3) for any $\varepsilon>0$, there exists $\delta>0$ such that $\omega(z,x)\le\delta$ and $\omega(z,y)\le\delta$ imply $d(x,y)\le\varepsilon$.

In [8], many examples and properties of W-distances were given.

Let X be a non-empty set and $\preccurlyeq$ a quasi-order (preorder or pseudo-order; that is, a reflexive and transitive relation) on X. Let $S(x)=\{y\in X\mid x\preccurlyeq y\}$ for $x\in X$, and $\le$ be the usual order in the extended real number system $[-\infty,\infty]$.

Let d be a quasi-metric (that is, not necessarily symmetric) on X. Then for the quasi-metric space (X,d), the concepts of W-distances, Cauchy sequences, completeness, and Banach contractions can be defined.

In a quasi-metric space (X,d) with a quasi-order $\preccurlyeq$, a set $S(u)$ for some $u\in X$ is said to be *$\preccurlyeq$-complete* if every nondecreasing Cauchy sequence in $S(u)$ converges. For details, see [20] and references therein.

Throughout this paper, let $\phi:X\times X\to(-\infty,\infty]$ be a function such that

(4) $\phi(x,z)\le\phi(x,y)+\phi(y,z)$ for any $x,y,z\in X$;
(5) $\phi(x,\cdot):X\to(-\infty,\infty]$ is lower semicontinuous for any $x\in X$; and
(6) there exists an $x_0\in X$ such that $\inf_{y\in X}\phi(x_0,y)>-\infty$.

The following is our main result in this paper:

Theorem 1. *Let (X,d) be a quasi-metric space. Let $\omega:X\times X\to[0,\infty)$ be a W-distance on X and $\phi:X\times X\to(-\infty,\infty]$ a function satisfying conditions* (4)–(6). *Define a quasi-order $\preccurlyeq$ on X by*

$$x\preccurlyeq y \quad \text{iff } x=y \text{ or } \phi(x,y)+\omega(x,y)\le 0.$$

Suppose that there exists a $u\in X$ such that $\inf_{y\in X}\phi(u,y)>-\infty$ and $S(u)=\{y\in X\mid u\preccurlyeq y\}$ is $\preccurlyeq$-complete.

Then the following equivalent statements hold:

(i) *There exists a maximal point $v\in S(u)$; that is,*

$$\forall w\in X\backslash\{v\},\quad \phi(v,w)+\omega(v,w)>0.$$

(ii) *If $T:S(u)\to 2^X$ satisfies the condition*

$$\forall x\in S(u)\backslash T(x)\quad \exists y\in X\backslash\{x\}\quad \text{such that } x\preccurlyeq y,$$

then T has a fixed point $v\in S(u)$; that is, $v\in T(v)$.

(iii) *A function* $f : S(u) \to X$ *satisfying* $x \preccurlyeq f(x)$ *for all* $x \in S(u)$ *has a fixed point.*
(iv) *If* $T : S(u) \to 2^X \setminus \{\emptyset\}$ *satisfies the condition*

$$\forall x \in S(u), \quad \forall y \in T(x), \quad x \preccurlyeq y \text{ holds},$$

then T *has a stationary point* $v \in S(u)$; *that is,* $T(v) = \{v\}$.
(v) *A family* $\mathscr{F}$ *of functions* $f : S(u) \to X$ *satisfying* $x \preccurlyeq f(x)$ *for all* $x \in S(u)$ *has a common fixed point* $v \in S(u)$.
(vi) *If* Y *is a subset of* X *such that for each* $x \in S(u) \setminus Y$ *there exists a* $z \in S(x) \setminus \{x\}$, *then there exists a* $v \in S(u) \cap Y$.

Proof. (i) By conditions (1) and (4), $\preccurlyeq$ is a quasi-order. We construct inductively a sequence of points $v_n \in S(u)$. To each v_n we let

$$S_n := \{v \in S(u) \mid v = v_n \text{ or } \phi(v_n, v) + \omega(v_n, v) \le 0\} = S(v_n)$$

and define the number

$$\gamma_n := \inf_{v \in S_n} \phi(v_n, v).$$

Note that each S_n is a closed subset of the $\preccurlyeq$-complete subset $S(u)$ since $\phi(v_n, \cdot) + \omega(v_n, \cdot)$ is l.s.c. by conditions (2) and (5).

Note that $v_n \in S_n \neq \emptyset$ and that $\gamma_n \le 0$. Let $u = v_0$. Then $S(u) = S_0$ and, by the hypothesis, $\gamma_0 \ge \inf_{v \in X} \phi(v_0, v) > -\infty$. Let $n \ge 1$ and assume that v_{n-1} with $\gamma_{n-1} > -\infty$ is already known. Then choose $v_n \in S_{n-1}$ such that

$$\phi(v_{n-1}, v_n) \le \gamma_{n-1} + \frac{1}{n}.$$

Since $v_n \in S_{n-1}$, for any $v \in S_n \setminus \{v_n\}$, we have

$$\begin{aligned} \phi(v_{n-1}, v) + \omega(v_{n-1}, v) &\le \phi(v_{n-1}, v_n) + \omega(v_{n-1}, v_n) + \phi(v_n, v) + \omega(v_n, v) \\ &\le \phi(v_n, v) + \omega(v_n, v) \le 0 \end{aligned}$$

and hence $v \in S_{n-1}$; that is, $S_{n-1} \supset S_n$. Therefore, we obtain

$$\begin{aligned} \gamma_n &= \inf_{v \in S_n} \phi(v_n, v) \ge \inf_{v \in S_n} (\phi(v_{n-1}, v) - \phi(v_{n-1}, v_n)) \\ &\ge \inf_{v \in S_{n-1}} (\phi(v_{n-1}, v) - \phi(v_{n-1}, v_n)) \\ &= \gamma_{n-1} - \phi(v_{n-1}, v_n) \ge -\frac{1}{n}. \end{aligned}$$

Therefore, for $v \in S_n \setminus \{v_n\}$, we have

$$\omega(v_n, v) \le -\phi(v_n, v) \le -\gamma_n \le \frac{1}{n}.$$

Since ω is a W-distance, for any $\varepsilon > 0$ we can choose a sufficiently large n such that

$$\omega(v_n, v) \le \frac{1}{n} \text{ and } \omega(v_n, v') \le \frac{1}{n} \text{ imply } d(v, v') < \varepsilon$$

for all $v, v' \in S_n$. Therefore the diameters of the sets S_n tend to zero. Moreover for all $k \ge n$ we have $v_k \in S_k \subset S_n$ and hence $d(v_n, v_k) \le 1/n$. Thus the sequence $\{v_n\}$ is Cauchy in the $\preccurlyeq$-complete set $S(u)$ and hence converges to some $v^* \in S(u)$. Clearly we have $v^* \in \bigcap_{n=0}^{\infty} S_n$. Since the diameters of the sets S_n tend to zero, we have $\bigcap_{n=0}^{\infty} S_n = \{v^*\}$. We claim that v^* is maximal. Otherwise, there exists a $w \in X \setminus \{v^*\}$ such that

$$\phi(v^*, w) + \omega(v^*, w) \le 0.$$

Since $v^* \in S_n$ for all n and hence

$$\phi(v_n, v^*) + \omega(v_n, v^*) \le 0$$

and

$$\phi(v_n, w) + \omega(v_n, w) \le 0$$

by the triangle inequalities of ϕ and ω. Therefore, $w \in S_n$ for all n. Therefore, we should have $w = v^*$. This contradiction completes our proof of (i).

(i) $\Rightarrow$ (ii) Suppose $v \notin T(v)$. Then there exists a $y \in X \setminus \{v\}$ such that $\phi(v, y) + \omega(v, y) \le 0$, which contradicts (i).

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Suppose that T has no stationary point; that is, $T(x) \setminus \{x\} \ne \emptyset$ for all $x \in S(u)$. Choose a choice function f on $\{T(x) \setminus \{x\} \mid x \in S(u)\}$. Then f has no fixed point by its definition. However, for any $x \in S(u)$, we have $x \ne f(x)$ and there exists a $y = f(x) \in T(x) \setminus \{x\}$ such that $x \preccurlyeq y = f(x)$. Therefore, by (iii), f has a fixed point, a contradiction.

(iv) $\Rightarrow$ (v) Define a function $T : S(u) \to 2^X$ by $T(x) := \{f(x) \mid f \in \mathscr{F}\}$ for all $x \in S(u)$. Since $x \preccurlyeq f(x)$ for all $x \in S(u)$ and all $f \in \mathscr{F}$, by (iv), T has a stationary point $v \in S(u)$, which is a common fixed point of $\mathscr{F}$.

(v) $\Rightarrow$ (i) Suppose that for any $x \in S(u)$, there exists a $y \in X \setminus \{x\}$ such that $x \preccurlyeq y$. Choose $f(x)$ to be one of such y. Then $f : S(u) \to X$ has no fixed point by its definition. However, $x \preccurlyeq f(x)$ for all $x \in S(u)$. Let $\mathscr{F} = \{f\}$. By (v), f has a fixed point, a contradiction.

(i) $\Rightarrow$ (vi) By (i), there exists a $v \in S(u)$ such that $\phi(v, w) + \omega(v, w) > 0$ for all $w \ne v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in S(u) \cap Y$.

(vi) $\Rightarrow$ (i) For all $x \in X$, let

$$A(x) := \{y \in X \mid x \ne y,\ \phi(x, y) + \omega(x, y) \le 0\} = S(x) \setminus \{x\}.$$

Choose $Y = \{x \in X \mid A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (vi) is satisfied. Therefore, by (vi), there exists a $v \in S(u) \cap Y$. Hence $A(v) = \emptyset$; that is, $\phi(v, w) + \omega(v, w) > 0$ for all $w \ne v$. Hence (i) holds.

This completes our proof. □

Theorem 1′. *Under the hypothesis of Theorem* 1, *the following also holds*:

(vii) *If, for each* $v \in S(u)$ *with* $\inf_{y\in X} \phi(v,y) < 0$, *there exists a* $w \in S(v)\backslash\{v\}$, *then there exists an* $x_0 \in S(u)$ *such that* $\inf_{y\in X} \phi(x_0,y) \geq 0$.

In fact, any of (i)–(vi) *implies* (vii). *Conversely,* (vii) *implies any of* (i)–(vi) *whenever either* (a) $\omega(x,y)=0$ *implies* $x=y$; *or* (b) $\phi(x,x)=0$ *for all* $x\in X$.

Proof. (i) ⇒ (vii) Suppose that for any $x \in S(u)$, we have $\inf_{y\in X} \phi(x,y) < 0$. Then there exists a $z \in S(x)\backslash\{x\}$; that is $x \prec z$. However, by (i), there exists a maximal element $v \in S(u)$; that is, $v \not\preccurlyeq w$ for all $w \in X\backslash\{v\}$. This is a contradiction.

(vii) ⇒ (i) Suppose that for each $v \in S(u)$, there exists a $w \in X\backslash\{v\}$ such that

$$\phi(v,w) + \omega(v,w) \leq 0 \quad \text{or} \quad w \in S(v)\backslash\{v\}.$$

Then by (vii), there exists an $x_0 \in S(u)$ such that $\inf_{y\in X} \phi(x_0,y) \geq 0$. For $x_0 = v$ and $x_1 = w$, we have

$$\phi(x_0,x_1) + \omega(x_0,x_1) \leq 0 \quad \text{or} \quad x_1 \in S(x_0)\backslash\{x_0\}.$$

Note that we should have $\inf_{y\in X} \phi(x_0,y) = 0$; otherwise, we have a contradiction

$$0 < \inf_{y\in X} \phi(x_0,y) + \omega(x_0,x_1) \leq 0.$$

Since $\omega(x_0,x_1) \geq 0$ and $\phi(x_0,x_1) \leq 0$, we should have $\phi(x_0,x_1) = \inf_{y\in X} \phi(x_0,y) = 0$ and hence $\omega(x_0,x_1) = 0$.

(a) If $\omega(x_0,x_1) = 0$ implies $x_0 = x_1$, then we have a contradiction.

(b) Similarly, for $x_1 = v \in S(u)$ and $x_2 = w$, we have

$$\phi(x_1,x_2) + \omega(x_1,x_2) \leq 0 \quad \text{or} \quad x_2 \in S(x_1)\backslash\{x_1\}.$$

Since $x_2 \in S(x_1)\backslash\{x_1\} \subset S(x_0)$, we have either

$$x_2 = x_0 \quad \text{or} \quad \phi(x_0,x_2) + \omega(x_0,x_2) \leq 0.$$

If $x_2 = x_0$, then by assumption (b), we have $\phi(x_0,x_2) = 0$ and $\omega(x_0,x_2) = 0$. If $\phi(x_0,x_2) + \omega(x_0,x_2) \leq 0$, $\inf_{y\in X} \phi(x_0,y) = 0$ implies $\phi(x_0,x_2) = 0$ and $\omega(x_0,x_2) = 0$ as above.

Now by condition (3), $\omega(x_0,x_1) = \omega(x_0,x_2) = 0$ implies $d(x_1,x_2) = 0$, or $x_1 = x_2$. This is a contradiction. This completes our proof. □

The following is a simplified form of Theorems 1 and 1′:

Theorem 2. *Let* (X,d) *be a complete quasi-metric space,* $\omega : X\times X \to [0,\infty)$ *a W-distance on* X, $\phi : X\times X \to (-\infty,\infty]$ *a function satisfying conditions* (4)–(6), *and* $x_0 \in X$ *satisfying* $\inf_{y\in X} \phi(x_0,y) > -\infty$ *in condition* (6). *Define a quasi-order* $\preccurlyeq$ *on* X *by*

$$x \preccurlyeq y \quad \textit{iff} \quad x = y \ \textit{or} \ \phi(x,y) + \omega(x,y) \leq 0.$$

Then the following holds:

(i) *There exists a maximal point* $v \in S(x_0)$; *that is,* $v \preccurlyeq w$ *implies* $v = w$.

(ii) *If* $T: X \to 2^X$ *satisfies*

$$\forall x \in X \backslash T(x) \quad \exists y \in X \backslash \{x\} \quad \text{such that } x \preccurlyeq y,$$

then T *has a fixed point.*

(iii) *A function* $f: X \to X$ *satisfying* $x \preccurlyeq f(x)$ *for all* $x \in X$ *has a fixed point.*

(iv) *If* $T: X \to 2^X \backslash \{\emptyset\}$ *satisfies*

$$\forall x \in X \quad \forall y \in T(x), \quad x \preccurlyeq y,$$

then T *has a stationary point.*

(v) *A family* $\mathscr{F}$ *of functions* $f: X \to X$ *such that* $x \preccurlyeq f(x)$ *for all* $x \in X$ *has a common fixed point.*

(vi) *If* Y *is a subset of* X *such that for each* $x \in X \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *such that* $x \preccurlyeq z$, *then we have* $S(x_0) \cap Y \neq \emptyset$.

(vii) *If, for each* $v \in X$ *with* $\inf_{y \in X} \phi(v, y) < 0$, *there exists a* $w \neq v$ *such that* $v \preccurlyeq w$, *then there exists an* $x_0 \in X$ *such that* $\inf_{y \in X} \phi(x_0, y) \geq 0$.

Proof. Since there exists an $x_0 \in X$ such that $\inf_{y \in X} \phi(x_0, y) > -\infty$ by (6), put $u = x_0$ and $S(u) = \{y \in X \mid u \preccurlyeq y\} = \{y \in X \mid y = u \text{ or } \phi(u, y) + \omega(u, y) \leq 0\}$. Since $\phi(u, \cdot) + \omega(u, \cdot)$ is l.s.c. by conditions (2) and (5), $S(u)$ is a closed subset of the complete space (X, d). Therefore, $S(u)$ is complete and hence $\preccurlyeq$-complete. Now, we can apply Theorems 1 and 1′.

From Theorem 2(i), we deduce the following well-known central result of Ekeland [6] on the variational principle for approximate solutions of minimization problems.

Theorem 3 (Ekeland [6]). *Let* (X, d) *be a complete quasi-metric space, and* $F: X \to (-\infty, \infty]$ *a proper lower semicontinuous function bounded from below. Let* $\varepsilon > 0$ *be given, and a point* $u \in X$ *such that* $F(u) \leq \inf_X F + \varepsilon$. *Then for any* $\lambda > 0$, *there exists a point* $v \in B(u, \lambda) = \{x \in X \mid d(u, x) \leq \lambda\}$ *such that*

$$F(v) \leq F(u) \text{ and } F(w) > F(v) - \varepsilon\lambda^{-1} d(v, w) \text{ for any } w \in X, \ w \neq v.$$

Proof. Let $\omega(x, y) = \varepsilon\lambda^{-1} d(x, y)$ and $\phi(x, y) = F(y) - F(x)$ for $x, y \in X$ (here we let $\phi(x, y) = 0$ if $F(x) = F(y) = \infty$). Then conditions (1)–(5) for ω and ϕ are clearly satisfied. For condition (6), let $x_0 = u$. Then $\inf_{y \in X} \phi(x_0, y) = \inf_X F - F(x_0) \geq -\varepsilon > -\infty$ by hypothesis.

Now all of the requirements of Theorem 2 are satisfied. Therefore, by Theorem 2(i), there exists a point $v \in S(u)$ such that

$$\phi(v, w) + \omega(v, w) > 0 \quad \text{for all } w \in X \backslash \{v\}.$$

It suffices to show that $F(v) \leq F(u)$ and $v \in B(u, \lambda)$. Since $v \in S(u)$, we have

$$\phi(u, v) + \omega(u, v) \leq 0.$$

Since $\omega(u,v)\geq 0$, we should have

$$0\geq\phi(u,v)=F(v)-F(u) \quad \text{or } F(v)\leq F(u).$$

Moreover,

$$\omega(u,v)=\varepsilon\lambda^{-1}d(u,v)\leq-\phi(u,v)=F(u)-F(v)\leq F(x_0)-\inf_X F\leq\varepsilon$$

by hypothesis, and hence

$$\varepsilon\lambda^{-1}d(u,v)\leq\varepsilon \quad \text{or } d(u,v)\leq\lambda.$$

This completes our proof. □

As a simple application of Theorem 2(iii), we extend a fixed point theorem of Downing and Kirk [3].

Theorem 4. *Let X and Z be complete quasi-metric spaces with W-distances ω_X and ω_Z, respectively, and $f:X\to X$ a function. Suppose there exist a closed map $g:X\to Z$ and a function $\phi:g(X)\times g(X)\to(-\infty,\infty]$ satisfying conditions* (4)–(6) *on $g(X)$ such that*

$$\phi(g(x),gf(x))+\max\{\omega_X(x,f(x)),\omega_Z(g(x),gf(x))\}\leq 0 \quad \textit{for } x\in X.$$

Then f has a fixed point.

Proof. It is easily checked that

$$\omega(x,y)=\max\{\omega_X(x,y),\omega_Z(g(x),g(y))\} \quad \text{for } x,y\in X$$

is a W-distance on X, and that the function ϕ' defined on $X\times X$ by

$$\phi'(x,y)=\phi(g(x),g(y)) \quad \text{for } x,y\in X$$

satisfies conditions (4)–(6). Hence, for each $x\in X$,

$$\phi'(x,f(x))+\omega(x,f(x))\leq 0 \quad \text{or } x\preccurlyeq f(x).$$

Therefore, f has a fixed point by Theorem 2(iii).

Note that Theorem 4 reduces to the theorem of Downing and Kirk [3] whenever ω_X and ω_Z are distances and $\phi(x,y)=F(y)-F(x)$ for a function F as in Theorem 3.

3. Historical remarks

(1) The primitive versions of Theorem 2 for $\omega=d$ and $\phi(x,y)=F(y)-F(x)$, where $F:X\to(-\infty,\infty]$ is a proper lower semicontinuous function on X bounded from below, were obtained by the following authors:

(i) Phelps [22] and Ekeland [4–6].

(ii) Tuy [24], Penot [21], and Mizoguchi and Takahashi [12].

(iii) Caristi et al. [2].
(iv) Maschler and Peleg [11].
(v) Kasahara [9].
(vi) Takahashi [23].

Each of those authors applied their results to various problems. Note that some of the proofs of the above results are simplified or replaced by constructive ones in this paper.

(2) The equivalency of Theorem 2(i)–(v) for primitive versions and their generalizations were given in [14–18].

(3) Quasi-metric versions of Theorem 1(i) and (iii) for primitive versions were due to Hicks [7] and Theorem 1(i)–(v) and Theorem $1'$(vii) to Park and Kang [20]. For some other related results, see [20].

(4) Oettli and Théra [13] obtained Theorem 2(vi) for $\omega = d$ and showed that (i), (ii), (vi), and (vii) are equivalent. In the same spirit, Blum and Oettli [1] gave a proof of Theorem 2(vii). Note that, in [1, 13], the authors assumed that the space is metric and that $\phi(x,x) = 0$ for all x in addition to conditions (4)–(6).

(5) The concept of W-distances was introduced by Kada et al. [8]. In fact, Theorem 2(vii) for $\phi(x,y) = F(y) - F(x)$ as above was proved in [8] as a generalization of earlier works of Ume [25], Kim et al. [10], and others. In [8], a number of contractive type fixed point theorems were shown to be consequences of Theorem 2(vii).

(6) Note that all of the above authors obtained particular forms of Theorems 1 and $1'$ or Theorem 2. Applications of Theorems 1 and 2 can be seen in the papers quoted above and references therein. Especially, for new applications of (vi) and (vii), see [13, 8].

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