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INVITED ARTICLE

ELEMENTS OF THE KKM THEORY FOR GENERALIZED CONVEX SPACES

SEHIE PARK

ABSTRACT. In the present paper, we introduce fundamental results in the KKM theory for G-convex spaces which are equivalent to the Brouwer theorem, the Sperner lemma, and the KKM theorem. Those results are all abstract versions of known corresponding ones for convex subsets of topological vector spaces. Some earlier applications of those results are indicated. Finally, we give a new proof of the Himmelberg fixed point theorem and G-convex space versions of the von Neumann type minimax theorem and the Nash equilibrium theorem as typical examples of applications of our theory.

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1. Introduction

It is well-known that the Brouwer fixed point theorem has numerous equivalent formulations and applications in various fields of mathematics such as topology, nonlinear analysis, equilibrium theory in economics, game theory, and others.

Theorem (Brouwer, 1912)**.** *Let $\Delta_n = v_0 v_1 \cdots v_n$ be an n-simplex. A continuous map $f : \Delta_n \to \Delta_n$ has a fixed point $x_0 = f(x_0) \in \Delta_n$.*

There are a large number of different proofs of the Brouwer theorem; for the literature, see [P7,8]. One of the earlier proofs of the Brouwer theorem was given by Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) [KKM] using the following [Sp]:

Lemma (Sperner, 1928)**.** *Let K be a simplicial subdivision of an n-simplex $v_0 v_1 \cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0} v_{i_1} \cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, the number assigned to u is one of the integers $i_0, i_1, \cdots, i_k$. Then the total number of those n-simplexes of K, whose vertices receive all $n+1$ integers $0, 1, \cdots, n$, is odd. In particular, there is at least one such n-simplex.*

For proofs of the Sperner lemma, see [I,Sp,Z]. The lemma was first applied to new proofs of the invariance theorems on dimensions and domains in [Sp], and subsequently, to obtain the following KKM principle ("closed" version) in [KKM]:

Theorem (KKM, 1929)**.** *Let $F_i\,(0 \le i \le n)$ be $n+1$ closed [resp. open] subsets of an n-simplex $v_0 v_1 \cdots v_n$. If*

$$v_{i_0} v_{i_1} \cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

holds for all faces $v_{i_0} v_{i_1} \cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, then $\bigcap_{i=0}^{n} F_i \neq \emptyset$.

The KKM principle provides the foundations for many of the modern essential results in diverse areas of mathematical sciences; see [P7,8]. In 1974, Yoseloff [Y] derived the Sperner lemma from the Brouwer theorem, and hence those are equivalent to the KKM principle.

Since 1961, Ky Fan had obtained generalizations of the KKM principle [F1,7] and applied them to various problems (see [F2-7]). Especially, he established an elementary but very basic "geometrical" lemma for multimaps and a minimax inequality. Later, Browder [Br] restated the lemma in the more convenient form of a fixed point theorem by means of the Brouwer theorem and the partition of unity argument. Those results are also known to be equivalent to the KKM principle. Along with such developments, there have appeared numerous generalizations of known results and new applications related to the so-called KKM map. For the literature, see [P7,8] and references therein.

In 1987, the "open" version of the KKM principle was due to Kim [Ki] and Shih and Tan [ST], and later Lassonde [L2] showed that the closed and open versions of the KKM principle can be derived from each other.

Let D be a nonempty set and X a convex subset of a vector space such that $D \subset X$. A multimap $F : D \multimap X$ is called a KKM map if

$$\mathrm{co}\, A \subset F(A) = \bigcup_{a \in A} F(a) \quad \text{for each} \quad A \in \langle D \rangle,$$

where co denotes the convex hull and $\langle D \rangle$ the set of all nonempty finite subsets of D.

The KKM theory, first called by the author [P4], is the study of KKM maps and their applications. At the beginning, the theory was mainly concerned with convex subsets of topological vector spaces. Later, it has been extended to convex space by Lassonde [L1], and to spaces having certain families of contractible subsets (simply, C-spaces or H-spaces) by Horvath [H1-4]. This line of generalizations of earlier works is followed by many authors; see [PK2,P7,8].

On the other hand, in [PK1,3-6], the author introduced a more general concept than C-spaces and basic properties of KKM maps for such spaces, which seem to be more adequate for various purposes. Actually our new concept of generalized convex spaces or G-convex spaces is a common generalization of many of known abstract convexities without any linear structure developed in connection mainly with the fixed point theory and the KKM theory.

In the present paper, we introduce fundamental results in the KKM theory for G-convex spaces which are equivalent to the Brouwer theorem, the Sperner lemma, and the KKM principle. Those results are all abstract versions of known corresponding ones for convex subsets of topological vector spaces. Some earlier applications of those results are indicated. Finally we show, as typical examples of applications, that the Himmelberg fixed point theorem directly follows from the open version of the KKM principle, and give G-convex space versions of the von Neumann minimax theorem and the Nash equilibrium theorem.

2. Generalized convex spaces

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ having $n+1$ elements, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Note that $\phi_A|_{\Delta_J}$ can be regarded as ϕ_J and we may assume $\Gamma(J) \subset \Gamma(A)$.

Here Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J := \text{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$.

In our previous works [PK1-5], we assumed that D is a nonempty subset of X in the definition of a G-convex space $(X, D; \Gamma)$. From now on, we adopt the above definition; that is, we do not assume $D \subset X$ in general. In case $X \supset D$, then $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$.

There are a lot of examples of G-convex spaces :

Examples 2.1. If X is a convex subset of a vector space, $D \subset X$, and X has a topology such that each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology, then $(X, D; \Gamma)$ becomes a convex space generalizing the one due to Lassonde [L1]. Note that any convex subset of a topological vector space is a convex space, but not conversely.

Examples 2.2. If $X = D$ and Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$), and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then $(X; \Gamma)$ becomes a C-space (or an H-space) due to Horvath [H1-4]; see also [PK2].

Examples 2.3. Other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, and so on. For the literature, see [PK1,3-6]. Recently, further examples of G-convex spaces were given by the author [P9] as follows: L-spaces and B'-simplicial convexity of Ben-El-Mechaiekh et al., continuous images of G-convex spaces, Verma's or Stachó's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, mc-spaces of Llinares, hyperconvex metric spaces due to Aroszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$; and for any subset $Y \subset X$, the *convex hull* of Y is defined by as follows :

$$\Gamma\text{-co}\, Y := \bigcap \{Z \subset X : Z \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } Y\}.$$

It is easily seen that $\Gamma\text{-co}\, Y = \bigcup\{\Gamma\text{-co}\, N : N \in \langle Y \rangle\}$.

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from X into the power set 2^Y of Y. $T(x)$ is called the *value* of T at $x \in X$ and $T^-(y) := \{x \in X : y \in T(x)\}$ the *fiber* of T at $y \in Y$. Let $T(A) := \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if

$$\Gamma_N \subset F(N) \quad \text{for each } N \in \langle D \rangle.$$

A subset A of a topological space X is said to be *compactly closed* [resp. *open*] in X if for every compact set $K \subset X$, the set $A \cap K$ is *closed* [resp. *open*] in K.

3. The KKM theorem and a matching theorem

The following is a KKM theorem for G-convex spaces :

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*

(1.1) *F has compactly closed [resp. open] values; and*
(1.2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property.

Further, if F has compactly closed values and if

(1.3) *$\bigcap_{z \in M} F(z)$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{z \in D} F(z) \neq \emptyset.$$

Proof. Let $A = \{a_0, a_1, \cdots, a_n\} \in \langle D \rangle$. Then there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A$ such that, for any $0 \le i_0 < i_1 < \cdots < i_k \le n$, we have

$$\phi_A(\mathrm{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\}) \cap \phi_A(\Delta_n).$$

Since F is a KKM map, it follows that

$$\begin{aligned}\mathrm{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\} &\subset \phi_A^-(\Gamma(\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\}) \cap \phi_A(\Delta_n)) \\ &\subset \bigcup_{j=0}^{k} \phi_A^-(F(a_{i_j}) \cap \phi_A(\Delta_n)).\end{aligned}$$

Since $F(a_{i_j}) \cap \phi_A(\Delta_n)$ is closed [resp. open] in the compact subset $\phi_A(\Delta_n)$ of Γ_A, $\phi_A^-(F(a_{i_j}) \cap \phi_A(\Delta_n))$ is closed [resp. open] in Δ_n. Note that $v_i \multimap \phi_A^-(F(a_i) \cap \phi_A(\Delta_n))$ is a KKM map on $\{v_0, v_1, \cdots, v_n\}$. Hence, by the KKM principle, we have

$$\bigcap_{i=0}^{n} \phi_A^-(F(a_i) \cap \phi_A(\Delta_n)) \neq \emptyset,$$

which readily implies $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$. This completes the proof of the first part.

Note that the second part follows immediately from the first. □

Remarks. (1) Condition (1.1) is actually satisfied if we assume that, for each $a \in D$ and $A \in \langle D \rangle$, $F(a) \cap \Gamma_A$ is closed [resp. open] in Γ_A. This is said by some authors that F has finitely closed [resp. open] values.

(2) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \mathrm{co}$, the convex hull, Theorem 1 reduces to the celebrated KKM principle [KKM]. This principle was first used in [KKM] to obtain one of the most direct proofs of the Brouwer theorem, and later applied to topological results on Euclidean spaces in [A, AH]; see [PJ].

(3) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 1 extends Fan's KKM lemma [F1]. Fan [F1] applied his lemma to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a

compact convex set into a uniform space. Later, Fan [F2] also applied his lemma to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff fixed point theorem.

(4) For another forms of the KKM theorem for convex spaces and their applications, see [P1,4-6]. A far-reaching generalized form of Theorem 1 and its equivalent formulations were already given by the author and Kim [PK1,3,4]. There are a lot of more general "compactness" or "coercivity" conditions than (1.3); see [PK3,4]. However, for the simplicity, we adopt only (1.3) and a more simple form for a singleton $M = \{z_0\}$; see Theorems 3 and 5-11.

From Theorem 1, we have the following generalization of the Alexandroff-Pasynkoff theorem [AP]:

For $D = \{a_0, a_1, \cdots, a_n\}$, we denote as follows:

$$A_0 := \{a_0, \cdots, a_{n-1}\},$$
$$A_i := \{a_0, \cdots, \widehat{a_{i-1}}, \cdots, a_n\}$$

for $1 \leq i \leq n$.

Theorem 2. *Let $(X, D; \Gamma)$ be a G-convex space such that D is as above and $T : D \multimap X$ a multimap with compactly closed [resp. open] values such that*

(2.1) $X = T(D)$ *and*

(2.2) $\Gamma_{A_i} \subset T(a_i)$ *for* $0 \leq i \leq n$.

Then $\bigcap_{i=0}^{n} T(a_i) \neq \emptyset$.

Proof. We show that T is a KKM map. Let $N \in \langle D \rangle$. If $N = D$, then $\Gamma_N \subset X = T(N)$ by (2.1). Suppose that $N \subsetneq D$. Then, by (2.2),

$$\Gamma_N \subset \Gamma_{A_j} \subset T(a_j) \quad \text{for some} \quad a_j \in N,$$

and hence

$$\Gamma_N \subset \bigcup \{T(a_i) : a_i \in N\} = T(N).$$

Now the conclusion follows from Theorem 1. □

Remarks. (1) For $X = D = \Delta_n$ and $\Gamma = \text{co}$, the closed version of Theorem 2 reduces to the Alexandroff-Pasynkoff theorem [AP], which was first applied to the essentiality of the identity map of a simplex.

(2) The Alexandroff-Pasynkoff theorem readily implies the Brouwer fixed point theorem, and hence Theorems 1 and 2 are equivalent to the KKM principle; see [PJ], where some fixed point theorems, intermediate value theorems, various non-retract theorems, and the non-contractibility of spheres are shown to be equivalent to the KKM principle.

From now on, we are mainly concerned with the closed version of the KKM principle. From Theorem 1, we have the following Ky Fan type matching theorem for open covers:

Theorem 3. *Let $(X, D; \Gamma)$ be a G-convex space and $S : D \multimap X$ a map such that*

(3.1) *$S(z)$ is compactly open for each $z \in D$;*
(3.2) *$S^{-}(y)$ is nonempty for each $y \in X$ (that is, S is surjective); and*
(3.3) *$X \backslash S(z_0)$ is compact for some $z_0 \in X$.*

Then there exists an $N \in \langle D \rangle$ such that

$$\Gamma_N \cap \bigcap_{z \in N} S(z) \neq \emptyset.$$

Proof. Let $F : D \multimap X$ be a map given by $F(z) := X \backslash S(z)$ for $z \in D$. Then condition (3.1) implies (1.1). Suppose, on the contrary to the conclusion, that for any $N \in \langle D \rangle$, we have $\Gamma_N \cap \bigcap_{z \in N} S(z) = \emptyset$; that is, $\Gamma_N \subset X \backslash \bigcap_{z \in N} S(z) = \bigcup_{z \in N}(X \backslash S(z)) = F(N)$, which implies (1.2). Moreover, (3.3) implies (1.3) with $M = \{z_0\}$. Therefore, by Theorem 1, there exists a $\widehat{y} \in \bigcap_{z \in D} F(z) = \bigcap_{z \in D}(X \backslash S(z))$. Hence $\widehat{y} \notin S(z)$ or $z \notin S^{-}(\widehat{y})$ for all $z \in D$. This violates condition (3.2). This completes our proof. $\square$

Remark. The origin of Theorem 3 goes back to Ky Fan [F6,7] for a convex set $X = D$. For applications, see also [P1-3].

Theorem 3 can be stated in its contrapositive form and in terms of the complement $F(x)$ of $S(x)$ in X. Then we obtain the closed version of Theorem 1 with $M = \{z_0\}$.

4. Another whole intersection theorem

From Theorem 1, we have another whole intersection property:

Theorem 4. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$, $G : X \multimap X$ maps such that*

(4.1) *F has compactly closed values;*
(4.2) *for each $x \in X$ and each $N \in \langle D \backslash F^-(x) \rangle$, we have $\Gamma_N \subset X \backslash G^-(x)$;*
(4.3) *$x \in G(x)$ for each $x \in X$; and*
(4.4) *$\bigcap_{z \in M} F(z)$ is compact for some $M \in \langle D \rangle$.*

Then

$$\bigcap_{z \in D} F(z) \neq \emptyset.$$

Proof. In view of Theorem 1, it suffices to show that F is a KKM map. Otherwise, there exists a subset $N \in \langle D \rangle$ such that $\Gamma_N \not\subset F(N)$; that is, there exists an $x \in \Gamma_N$ such that $x \notin F(z)$ for all $z \in N$. Hence, $N \in \langle D \backslash F^-(x) \rangle$ and, by (4.2), we have $\Gamma_N \subset X \backslash G^-(x)$. Therefore, $x \in X \backslash G^-(x)$ or $x \notin G^-(x)$, which contradicts (4.3). $\square$

Remark. The first particular form of Theorem 4 is due to Tarafdar [Tr2] for a convex space $X = D$. Another forms of Theorem 4 also appear in [H1,P5] and others.

From now on, we consider the case $M = \{z_0\}$ for simplicity.

5. Geometric or section properties

In this section, we deduce two geometric forms of the whole intersection property (Theorem 4). The following is usually called the section property:

Theorem 5. *Let $(X, D; \Gamma)$ be a G-convex space and $A \subset D \times X$, $B \subset X \times X$ such that*

(5.1) *$\{y \in X : (z, y) \in A\}$ is compactly closed for each $z \in D$;*
(5.2) *for each $y \in X$ and each nonempty finite subset $N \subset \{z \in D : (x, y) \notin A\}$, we have $\Gamma_N \subset \{x \in X : (x, y) \notin B\}$;*
(5.3) *$(x, x) \in B$ for each $x \in X$; and*
(5.4) *$\{y \in X : (z_0, y) \in A\}$ is compact for some $z_0 \in D$.*

Then there exists an $x_0 \in X$ such that $D \times \{x_0\} \subset A$.

Proof of Theorem 5 using Theorem 4. For each $z \in D$, let $F(z) := \{y \in X : (z, y) \in A\}$. Then (5.1) $\Longrightarrow$ (4.1). Moreover, for each $x \in X$, let

$G(x) := \{y \in X : (x, y) \in B\}$. Then (5.2) $\Longrightarrow$ (4.2). Further (5.3) $\Longrightarrow$ (4.3) and (5.4) $\Longrightarrow$ (4.4). Therefore, by Theorem 4, we have

$$\bigcap_{z \in D} F(z) = \bigcap_{z \in D} \{y \in X : (z, y) \in A\} \neq \emptyset.$$

Hence there exists an $x_0 \in X$ such that $(z, x_0) \in A$ for all $z \in D$; that is, $D \times \{x_0\} \subset A$. This completes our proof. □

Remark. If $X = D$ is a convex subset of a topological vector space and if $A = B$, Theorem 5 reduces to Fan's 1961 Lemma [F1, Lemma 4]. He obtained his result from his own generalization of the KKM principle and applied it to a direct proof of the Tychonoff fixed point theorem. Other interesting applications of his useful lemma to fixed points, minimax theorems, equilibrium points, extension of monotone sets, potential theory, etc. have been made by Fan [F3] and many others; see [P7,8].

The following form of the geometric property is equivalent to Theorem 5 :

Theorem 6. *Let $(X, D; \Gamma)$ be a G-convex space and $A \subset D \times X$, $B \subset X \times X$ such that*

(6.1) *$\{y \in X : (z, y) \in A\}$ is compactly open for each $z \in D$;*
(6.2) *for each $y \in X$ and each nonempty finite subset $N \subset \{z \in D : (z, y) \in A\}$, we have $\Gamma_N \subset \{x \in X : (x, y) \in B\}$;*
(6.3) *for each $y \in X$, there exists a $z \in D$ such that $(z, y) \in A$; and*
(6.4) *$\{y \in X : (z_0, y) \notin A\}$ is compact for some $z_0 \in D$.*

Then there exists an $x_0 \in X$ such that $(x_0, x_0) \in B$.

Proof of Theorem 6 using Theorem 5. Consider Theorem 5 replacing (A, B) by their respective complements (A^c, B^c). Then (5.1) and (5.2) are satisfied by (6.1) and (6.2). Since (6.3) is the negation of the conclusion of Theorem 5, we should have the negation of (5.3). Therefore, the conclusion follows. □

Proof of Theorem 5 using Theorem 6. Similar. □

Remark. If $X = D$ is a convex subset of a topological vector space and if $A = B$, Theorem 6 reduces to Fan [F5, Theorem 2]. In this case, (6.2) merely tells that $\{x \in X : (x, y) \in A\}$ is convex.

6. The Fan-Browder type fixed point theorems

Theorem 6 can be easily reformulated to the following fixed point theorem :

Theorem 7. *Let* $(X, D; \Gamma)$ *be a* G*-convex space and* $S : D \multimap X$, $T : X \multimap X$ *two maps such that*

(7.1) $S(z)$ *is compactly open for each* $z \in D$*;*
(7.2) *for each* $y \in X$, $N \in \langle S^{-}(y) \rangle$ *implies* $\Gamma_N \subset T^{-}(y)$*;*
(7.3) *for each* $y \in X$, $S^{-}(y) \neq \emptyset$*; and*
(7.4) $X \backslash S(z_0)$ *is compact for some* $z_0 \in D$*.*

Then T *has a fixed point* $x_0 \in X$*; that is* $x_0 \in T(x_0)$*.*

Proof of Theorem 7 using Theorem 6. Let A and B be the graphs of S and T, respectively. Then conditions (7.1) - (7.4) imply (6.1) - (6.4). Therefore, by Theorem 6, there exists an $x_0 \in X$ such that $(x_0, x_0) \in B$, that is, T has a fixed point $x_0 \in X$. □

Proof of Theorem 6 using Theorem 7. Define $S(z) := \{y \in X : (z, y) \in A\}$ and $T(x) := \{y \in X : (x, y) \in B\}$. Apply Theorem 7. □

Corollary 7.1. *Let* $(X, D; \Gamma)$ *be a compact* G*-convex space and* $S : X \multimap D$, $T : X \multimap X$ *two maps such that*

(1) *for each* $x \in X$, $N \in \langle S(x) \rangle$ *implies* $\Gamma_N \subset T(x)$*; and*
(2) $X = \bigcup \{\operatorname{Int} S^{-}(z) : z \in D\}$*.*

Then T *has a fixed point* $x_0 \in X$*.*

Proof. Replacing S and T in Theorem 7 by $\operatorname{Int} S^{-}$ and T^{-}, respectively, observe the following:

(7.1) $\operatorname{Int} S^{-}(z)$ is open for each $z \in D$;
(7.2) for each $y \in X$, $N \in \langle (\operatorname{Int} S^{-})^{-}(y) \rangle \subset \langle S(y) \rangle$ implies $\Gamma_N \subset T(y)$ by (1);
(7.3) for each $y \in X$, by (2), there exists a $z \in D$ such that $z \in \operatorname{Int} S^{-}(z)$, and hence $(\operatorname{Int} S^{-})^{-}(y) \neq \emptyset$; and
(7.4) since X itself is compact, $X \backslash \operatorname{Int} S^{-}(z_0)$ is compact for any $z_0 \in D$.

Therefore, by Theorem 7, T^{-} has a fixed point $x_0 \in X$; that is, $x_0 \in T^{-}(x_0)$ or $x_0 \in T(x_0)$. This completes the proof. □

Corollary 7.2. *Let* $(X \supset D; \Gamma)$ *be a compact* G*-convex space and* $S : X \multimap D$ *a map such that*

(1) *for each* $x \in X$, $S(x)$ *is nonempty; and*
(2) *for each* $z \in D$, $S^{-}(z)$ *is open.*

Then there exists an $\widehat{x} \in X$ *such that* $\widehat{x} \in \Gamma\text{-co}\, S(\widehat{x})$.

The following simplified form of Corollary 7.1 or 7.2 is also a Fan-Browder type fixed point theorem :

Corollary 7.3. *Let* $(X; \Gamma)$ *be a compact* G*-convex space and* $T : X \multimap X$ *a map such that*

(1) *for each* $x \in X, T(x)$ *is* Γ*-convex; and*
(2) $X = \bigcup\{\text{Int}\, T^{-}(y) : y \in X\}$.

Then T *has a fixed point.*

Proof. Replacing $(S^{-}, \Gamma\text{-co}\, S)$ in Corollary 7.2 by $(\text{Int}\, T^{-},\ T)$, we have the conclusion immediately. □

Remarks. (1) For a convex subset X of a topological vector space E, if $T^{-}(y)$ itself is open, then Corollary 7.3 reduces to Browder's fixed point theorem [Br]. Condition (2) was first considered by Tarafdar [Tr1].

(2) Note that Browder's result is a reformulation of Fan's geometric lemma [F1] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it is known as the Fan-Browder fixed point theorem.

(3) Browder [Br] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. For further developments on generalizations and applications of the Fan-Browder theorem, we refer to Park [P1,5].

7. The existence theorems of maximal elements

Any binary relation R in a set X can be regarded as a multimap $T : X \multimap X$ and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a multimap if $T(x_0) = \emptyset$.

The Fan-Browder type fixed point theorem is used by Borglin and Keiding [BK] and Yannelis and Prabhakar [YP] to the existence of maximal elements in mathematical economics.

In this section, we derive some results on maximal elements as follows :

Theorem 8. *Let $(X, D; \Gamma)$ be a G-convex space and $S : X \multimap D$, $T : X \multimap X$ two maps such that*

(8.1) *$S^-(z)$ is compactly open for each $z \in D$;*
(8.2) *for each $x \in X$, $N \in \langle S(x) \rangle$ implies $\Gamma_N \subset T(x)$;*
(8.3) *for each $x \in X$, $x \notin T(x)$; and*
(8.4) *$X \backslash S^-(z_0)$ is compact for some $z_0 \in D$.*

Then there exists an $\widehat{x} \in X$ such that $S(\widehat{x}) = \emptyset$.

Proof of Theorem 8 using Theorem 7. Suppose that $Sx \neq \emptyset$ for each $x \in X$. Consider Theorem 7 replacing S, T by S^-, T^-, respectively. Then all of the requirements of Theorem 7 are satisfied. Therefore, there exists an $x_0 \in X$ such that $x_0 \in T^-(x_0)$ or $x_0 \in T(x_0)$. But this violates (8.3). This completes our proof. □

Proof of Theorem 7 using Theorem 8. Replacing S, T in Theorem 7 by S^-, T^-, respectively, follow the above proof. □

Corollary 7.2 is equivalent to the following simple consequence of Theorem 8.

Corollary 8.1. *Let $(X \supset D; \Gamma)$ be a compact G-convex space and $S : X \multimap D$ a map such that*

(1) *$x \notin \Gamma\text{-co}S(x)$ for $x \in X$; and*
(2) *$S^-(z)$ is open for each $z \in D$.*

Then there exists an $\widehat{x} \in X$ such that $S(\widehat{x}) = \emptyset$.

8. Analytic alternatives

From Theorem 6, we have the following analytic alternative which is equivalent to Theorem 4:

Theorem 9. *Let $(X, D; \Gamma)$ be a G-convex space and $A, B \subset C$ sets. Let $f : D \times X \to C$ and $g : X \times X \to C$ be functions satisfying*

(9.1) *$\{y \in X : f(z, y) \in A\}$ is compactly open for each $z \in D$;*

(9.2) *for each* $y \in X$, $N \in \langle\{z \in D : f(z,y) \in A\}\rangle$ *implies* $\Gamma_N \subset \{x \in X : g(x,y) \in B\}$; *and*
(9.3) $\{y \in X : f(z_0, y) \notin A\}$ *is compact for some* $z_0 \in D$.

Then either

(a) *there exists a* $\widehat{y} \in X$ *such that* $f(z, \widehat{y}) \notin A$ *for all* $z \in D$; *or*
(b) *there exists a* $\widehat{x} \in X$ *such that* $g(\widehat{x}, \widehat{x}) \in B$.

Proof. It is immediate that Theorem 9 follows from Theorem 6 by replacing A, B by

$$A' = \{(z,y) \in D \times X : f(z,y) \in A\}, \quad B' = \{(x,y) \in X \times X : g(x,y) \in B\},$$

respectively. Similarly, we can obtain the converse. □

Remark. The first form of Theorem 9 is due to Lassonde [L1]. For another form, see [P5,PK4].

From Theorem 9, we obtain the following analytic alternative which is a basis of various equilibrium problems:

Theorem 10. *Let* $(X, D; \Gamma)$ *be a G-convex space,* $f : D \times X \to \overline{\mathbb{R}}$ *and* $g : X \times X \to \overline{\mathbb{R}}$ *two extended real valued functions, and* $\alpha, \beta \in \overline{\mathbb{R}}$. *Suppose that*

(10.1) $\{y \in X : f(z,y) > \alpha\}$ *is compactly open for each* $z \in D$;
(10.2) *for each* $y \in X$, $N \in \langle\{z \in D : f(z,y) > \alpha\}\rangle$ *implies* $\Gamma_N \subset \{x \in X : g(x,y) > \beta\}$; *and*
(10.3) $\{y \in X : f(z_0, y) \le \alpha\}$ *is compact for some* $z_0 \in D$.

Then either

(a) *there exists a* $\widehat{y} \in X$ *such that* $f(z, \widehat{y}) \le \alpha$ *for all* $z \in D$; *or*
(b) *there exists a* $\widehat{x} \in X$ *such that* $g(\widehat{x}, \widehat{x}) > \beta$.

Proof. Put $C = \overline{\mathbb{R}}$, $A = (\alpha, \infty]$, and $B = (\beta, \infty]$ in Theorem 9. □

Remark. If $X = D$ is a compact convex space, Theorem 10 reduces to the principal result of Ben-El-Mechaiekh et al. [BDG1,2], where this result is applied to variational inequalities of Hartman-Stampacchia and Browder and a generalization of the Ky Fan minimax inequality.

9. Minimax inequalities

From Theorem 10, we immediately have the following generalized form of the Ky Fan minimax inequality [F5] :

Theorem 11. *Under the hypothesis of Theorem 10, if* $\alpha = \beta = \sup\{g(x,x) : x \in X\}$*, then*

(c) *there exists a* $\widehat{y} \in X$ *such that*

$$f(z,\widehat{y}) \leq \sup_{x\in X} g(x,x) \quad \text{for all } z \in D; \text{ and}$$

(d) *we have the minimax inequality*

$$\inf_{y\in X} \sup_{z\in D} f(z,y) \leq \sup_{x\in X} g(x,x).$$

In order to show Theorem 11 is equivalent to any of Theorems 4 - 10, we give the following :

Proof of Theorem 4 using Theorem 11. Define functions $f : D \times X \to \mathbb{R}$ and $g : X \times X \to \mathbb{R}$ by

$$f(z,y) = \begin{cases} 0 & \text{if } y \in F(z) \\ 1 & \text{otherwise} \end{cases}$$

for $(z,y) \in D \times X$ and

$$g(x,y) = \begin{cases} 0 & \text{if } y \in G(x) \\ 1 & \text{otherwise} \end{cases}$$

for $(x,y) \in X \times X$. Then $\alpha = \beta = 0$ by (4.3). Note that, for each $z \in D$, since $\{y \in X : f(z,y) > 0\} = \{y \in X : y \notin F(z)\} = X \backslash F(z)$ is compactly open by (4.1), we have (10.1). And, since $F(z_0) = \{y \in X : f(z_0,y) \leq 0\}$ is compact for some $z_0 \in D$, we have (10.3). Moreover, (4.2) implies (10.2). Therefore, by Theorem 11(c), there exists a $\widehat{y} \in X$ such that

$$f(z,\widehat{y}) \leq \sup_{x\in X} g(x,x) = 0 \quad \text{for all } z \in D;$$

whence $\widehat{y} \in F(z)$ for all $z \in D$. This completes our proof of Theorem 4. □

Therefore, we have the following :

Proposition 1. *Each of Theorems 4 - 11 is equivalent to the other, and follows from any of Theorems 1 - 3.*

For a G-convex space $(X \supset D; \Gamma)$, the closed version of the KKM theorem (Theorem 1) can also be reformulated to another minimax inequality as follows:

Theorem 12. *Let $(X \supset D; \Gamma)$ be a G-convex space, $\phi : D \times X \to \overline{\mathbb{R}}$ an extended real valued function, and $\gamma \in \overline{\mathbb{R}}$ such that*

(12.1) $\{y \in X : \phi(z, y) \leq \gamma\}$ *is compactly closed for each $z \in D$;*

(12.2) *for each $N \in \langle D \rangle$ and for each $y \in \Gamma_N$, we have $\min_{z \in N} \phi(z, y) \leq \gamma$; and*

(12.3) $\{y \in X : \phi(z_0, y) \leq \gamma\}$ *is compact for some $z_0 \in D$.*

Then (a) *there exists a $\widehat{y} \in X$ such that*

$$\phi(z, \widehat{y}) \leq \gamma \quad \text{for all} \quad z \in D;$$

and (b) *if $\gamma = \sup_{x \in D} \phi(x, x)$, then we have the minimax inequality:*

$$\min_{y \in X} \sup_{z \in D} \phi(z, y) \leq \sup_{x \in D} \phi(x, x).$$

Proof of Theorem 12 using Theorem 1. Let $F(z) := \{y \in X : \phi(z, y) \leq \gamma\}$ for $z \in D$. Then, by (12.1) and (12.3), $F : D \multimap X$ has compactly closed values and $F(z_0)$ is compact for some $z_0 \in D$. Hence, conditions (1.1) and (1.3) are satisfied. Moreover, by (12.2), F is a KKM map: Indeed, suppose that there exists an $N \in \langle D \rangle$ such that $\Gamma_N \not\subset F(N)$. Choose a $y \in \Gamma_N$ such that $y \notin F(N)$; that is, $y \notin F(z)$ or $\phi(z, y) > \gamma$ for all $z \in N$. Then $\min_{z \in N} \phi(z, y) > \gamma$, contradicting (12.2). Therefore, by Theorem 1, there exists a $\widehat{y} \in X$ such that $\widehat{y} \in \bigcap_{z \in D} F(z) \neq \emptyset$; that is, $\phi(z, \widehat{y}) \leq \gamma$ for all $z \in D$. This completes the proof of (a). Note that (b) immediately follows from (a). □

Proof of Theorem 1 for $(X \supset D; \Gamma)$ using Theorem 12. Define $\phi : D \times X \to \mathbb{R}$ by

$$\phi(z, y) = \begin{cases} 0 & \text{if } y \in F(z) \\ 1 & \text{otherwise} \end{cases}$$

for $(z, y) \in D \times X$ and put $\gamma = 0$ in Theorem 12. Since $\{y \in X : \phi(z, y) \leq 0\} = F(z)$, condition (1.1) implies (12.1) and condition (1.3) for $M = \{z_0\}$

implies (12.3). [Note that we are still working on the particular case of (1.3) for a singleton M.] Moreover, since F is a KKM map by (1.2), condition (12.2) follows: Indeed, suppose that there exist an $N \in \langle D \rangle$ and a $y \in \Gamma_N$ such that $\min_{z \in N} \phi(z, y) > 0$. Then $y \notin F(z)$ for all $z \in N$; that is, $y \in \Gamma_N \not\subset F(N)$, a contradiction. Therefore, by Theorem 12, there exists a $\widehat{y} \in X$ such that $\phi(z, \widehat{y}) = 0$ for all $z \in D$; that is, $\widehat{y} \in \bigcap_{z \in D} F(z)$. This completes our proof of the closed version of Theorem 1. □

Remark. The first particular form of Theorem 12 is due to Zhou and Chen [ZC], who applied it to a variation of the Ky Fan minimax inequality, a saddle point theorem, and a quasi-variational inequality.

Proposition 2. *For a G-convex space $(X \supset D; \Gamma)$, Theorem 4 implies the KKM priciple, and, consequently, all of Theorems 1 - 12 are mutually equivalent.*

Proof. Consider the particular case where $\Gamma_N = \Gamma\text{-co}\, N$ for all $N \in \langle D \rangle$. Under the hypothesis of the closed version of Theorem 1, let $G : X \multimap X$ be defined by

$$X \backslash G^-(x) := \Gamma\text{-co}(D \backslash F^-(x)) \quad \text{for } x \in X.$$

Then clearly (4.2) holds. We claim that (4.3) holds. Suppose, on the contrary, that $x \notin G(x)$ for some $x \in X$. Then $x \notin G^-(x)$ and hence $x \in X \backslash G^-(x)$. This implies $x \in \Gamma_N$ for some $N \in \langle D \backslash F^-(x) \rangle$ by the definition of G. Then, for all $z \in N$, we have $z \in D \backslash F^-(x) \iff z \notin F^-(x) \iff x \notin F(z)$ and hence $x \notin F(N)$. Therefore $\Gamma_N \not\subset F(N)$, which violates (1.2). Now all of the requirements of Theorem 4 are satisfied, and hence the closed version of Theorem 1 follows from Theorem 4. Therefore, the KKM principle follows from Theorem 4. Further, in view of Proposition 1, each of Theorems 1 - 11 is equivalent to the KKM principle. Moreover, since Theorems 1 and 12 are equivalent, we have the conclusion. □

Recall that an extended real valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

For a G-convex space $(X; \Gamma)$, a real function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

From now on, we mainly consider compact G-convex spaces $(X;\Gamma)$ for simplicity.

Theorem 13. *Let $(X;\Gamma)$ be a compact G-convex space and $f, g : X \times X \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(13.1) $f(x,y) \leq g(x,y)$ *for each* $(x,y) \in X \times X$,
(13.2) *for each* $x \in X$, $g(x,\cdot)$ *is quasiconcave on* X*; and*
(13.3) *for each* $y \in X$, $f(\cdot,y)$ *is l.s.c. on* X.

Then we have

$$\min_{y\in X}\sup_{x\in X} f(x,y) \leq \sup_{x\in X} g(x,x).$$

Proof. Observe that $\sup_{x\in X} f(x,y)$ is by (13.3) a l.s.c. function of y on the compact space X, and therefore its minimum exists. If $\sup_{x\in X} g(x,x) = +\infty$, then the inequality in the conclusion holds automatically. If $\alpha = \beta = \sup_{x\in X} g(x,x) < +\infty$, then by Theorem 11, we have the conclusion. □

Remarks. (1) For $f = g$, Theorem 13 reduces to Fan's minimax inequality [F5]. Fan obtained his inequality from his own generalization of the KKM principle, and applied it to deduce fixed point theorems, theorems on sets with convex sections, a fundamental existence theorem in potential theory, and so on.

(2) Later, the inequality has been an important tool in nonlinear analysis, game theory, and economic theory; see [P7,8].

In particular, we have the following :

Corollary 13.1. *Under the hypothesis of Theorem 13, if $g(x,x) \leq 0$ for all $x \in X$, then there exists a $y_0 \in X$ such that $f(x,y_0) \leq 0$ for all $x \in X$. Thus in particular*

$$\min_{y\in X}\sup_{x\in X} f(x,y) \leq 0.$$

10. Variational inequalities

Theorem 13 can be applied to the existence of solutions of certain variational inequalities:

Theorem 14. *Let* $(X;\Gamma)$ *be a compact* G*-convex space and* $p,q : X\times X \to \mathbb{R}$ *and* $h : X \to \mathbb{R}$ *functions satisfying*

(14.1) $p(x,y) \leq q(x,y)$ *for each* $(x,y) \in X \times X$*, and* $q(x,x) \leq 0$ *for all* $x \in X$*;*
(14.2) *for each* $x \in X$*,* $q(x,\cdot) + h(\cdot)$ *is quasiconcave on* X*; and*
(14.3) *for each* $y \in X$*,* $p(\cdot,y) - h(\cdot)$ *is l.s.c. on* X*.*

Then there exists a $y_0 \in X$ *such that*

$$p(x,y_0) + h(y_0) \leq h(x) \quad \textit{for all} \ \ x \in X.$$

Proof. Let

$$f(x,y) := p(x,y) + h(y) - h(x) \ \text{ and } \ g(x,y) := q(x,y) + h(y) - h(x)$$

for $(x,y) \in X \times Y$. Then f and g satisfy the requirements of Theorem 13. Furthermore, $g(x,x) = q(x,x) \leq 0$ for
all $x \in X$. Therefore, by Corollary 13.1, the conclusion follows. □

Remarks. (1) Putting $h = 0$, Theorem 14 reduces to Corollary 13.1.

(2) Theorem 14 is a basis of existence theorems of many results concerning variational inequalities; see [P5] and references therein.

Theorem 15. *Let* $(X;\Gamma)$ *be a compact* G*-convex space and* $p,q : X\times X \to \mathbb{R}$ *functions such that*

(15.1) $p \leq q$ *on the diagonal* $\Delta := \{(x,x) : x \in X\}$ *and* $q \leq p$ *on* $(X \times X)\backslash\Delta$*;*
(15.2) *for each* $x \in X$*,* $y \mapsto q(y,y) - q(x,y)$ *is quasiconcave on* X*; and*
(15.3) *for each* $y \in X$*,* $x \mapsto p(x,y)$ *is u.s.c. on* X*.*

Then there exists a $y_0 \in X$ *such that*

$$p(y_0,y_0) \leq p(x,y_0) \quad \textit{for all} \ \ x \in X.$$

Proof. Define $f,g : X \times X \to \mathbb{R}$ by

$$\begin{aligned} f(x,y) &:= p(y,y) - p(x,y) \\ g(x,y) &:= q(y,y) - q(x,y) \end{aligned}$$

Then f and g satisfy the hypothesis of Theorem 13. Since $g(x,x)=0$ for all $x \in X$, Corollary 13.1 implies that $f(x,y_0) \leq 0$ for all $x \in X$. This implies the conclusion. □

Remark. For a convex space X and $p=q$, Theorem 15 reduces to Fan [F5], which was shown to be very useful in nonlinear functional analysis. In fact, the Tychonoff (and hence, the Brouwer) fixed point theorem, Browder's variational inequality, and many other applications follow from his result.

Since Theorem 15 implies the Brouwer fixed point theorem, we have the following:

Proposition 3. *For a compact G-convex space $(X;\Gamma)$, Theorems 13 - 15 follow from any of Theorems 1 - 12, and each of Theorems 1 - 15 and Corollaries is equivalent to the KKM principle.*

11. Best approximations

A simple consequence of Theorem 15 is the following well-known existence result on best approximations originated from Ky Fan [F4] :

Theorem 16. *Let X be a compact convex subset of a topological vector space E and $f : X \to E$ a continuous function. Then for any continuous seminorm p on E, there exists a point $y_0 \in X$ such that*

$$p(y_0 - f(y_0)) \leq p(x - f(y_0)) \quad \text{for all} \quad x \in X.$$

Proof. For each $y \in X$, $x \mapsto p(y-f(y)) - p(x-f(y))$ is convex on X, and for each $x \in X$, $y \mapsto p(x-f(y))$ is continuous. Therefore, by Theorem 15, we have a $y_0 \in X$ satisfying the conclusion. □

Remark. Further if E is a normed vector space and p is a norm on E, then Theorem 16 reduces to the well-known existence result on best approximation due to Ky Fan [F4, Theorem 2], which immediately implies the Schauder fixed point theorem; that is, the normed space version of the Brouwer theorem. Therefore, Theorem 16 generalizes and implies the Brouwer theorem.

12. Fixed point theorems

In this section, we show that the open version of the KKM theorem is also useful to deduce very general fixed point theorems for topological vector spaces or G-convex spaces. For simplicity, we give only one example.

Recall that a multimap $F : X \multimap Y$, where X and Y are topological spaces, is said to be *upper semicontinuous* (u.s.c.) whenever $\{x \in X : F(x) \cap C \neq \emptyset\}$ is closed in X for each closed subset C of Y; and *compact* if the range $F(X)$ is contained in a compact subset of Y.

We give a simple proof of the following in [Hi]:

Theorem 17 (Himmelberg, 1972). *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $T : X \multimap X$ a compact u.s.c. multimap with nonempty closed convex values. Then T has a fixed point $x_0 \in T(x_0)$.*

Proof. Let U be a convex neighborhood of the origin 0 of E. Then there exists an open neighborhood V of 0 such that $\overline{V} \subset U$. Since $K := \overline{T(X)}$ is compact in X, there exists a finite subset $D := \{x_1, \cdots, x_n\} \subset K \subset X$ such that $K \subset \bigcup_{i=1}^n (x_i + V)$. Then $(X \supset D; \mathrm{co})$ is a G-convex space. For each i, let

$$F(x_i) := \{x \in X : T(x) \cap (x_i + \overline{V}) = \emptyset\}.$$

Since T is upper semicontinuous, each $F(x_i)$ is open in X. Moreover we have

$$\bigcap_{i=1}^{n} F(x_i) = \{x \in X : T(x) \cap \bigcup_{i=1}^{n} (x_i + \overline{V}) = \emptyset\} = \emptyset,$$

since $T(X) \subset K \subset \bigcup_{i=1}^n (x_i + V)$.

Now we apply the open version of Theorem 1. Since its conclusion does not hold, $F : D \multimap X$ can not be a KKM map. That is, there exist a subset $\{x_{i_1}, \cdots, x_{i_k}\} \subset D$ and an $x_U \in \mathrm{co}\{x_{i_1}, \cdots, x_{i_k}\}$ such that $x_U \notin \bigcup_{j=1}^k F(x_{i_j})$. Hence $T(x_U) \cap (x_{i_j} + \overline{V}) \neq \emptyset$ for each j; and note that $x_{i_j} + \overline{V} \subset x_{i_j} + U$. Let L be the subspace of E generated by D, and

$$M := \{y \in L : T(x_U) \cap (y + U) \neq \emptyset\}.$$

Since $T(x_U) \cap (x_{i_j} + U) \neq \emptyset$, we have $x_{i_j} \in M$ for all $j = 1, \cdots, k$. Since $L, T(x_U)$, and U are all convex, it is easily checked that M is convex. Therefore, $x_U \in M$ and, by definition of M, we get $T(x_U) \cap (x_U + U) \neq \emptyset$.

So, for each neighborhood U of 0, there exist $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(X)$ is relatively compact, we may assume that the net $\{y_U\}$ converges to some $x_0 \in K$. Since E is Hausdorff, the net $\{x_U\}$ also converges to x_0. Because T is upper semicontinuous with closed values, the graph of T is closed in $X \times T(X)$ and hence we have $x_0 \in T(x_0)$. This completes our proof. □

Remarks. (1) Theorem 17 includes the Brouwer fixed point theorem and its generalizations due to Schauder, Tychonoff, Hukuhara, Kakutani, Bohnenblust and Karlin, Fan, and Glicksberg; for the literature, see [P7,8].

(2) In forthcoming works of the author, it will be shown that we can derive much more general results than Theorem 17 from Theorem 1.

Since the KKM Theorem 1 is implied by the Brouwer theorem, we can conclude the following:

Proposition 4. *Any of Theorems 1 - 17 and Corollaries is equivalent to the Brouwer fixed point theorem.*

Therefore, the Brouwer theorem, the Sperner lemma, Theorems 1 - 17 and Corollaries are all equivalent and can be regarded as the fundamental results in the KKM theory for generalized convex spaces. Note that the original version of each of them is also equivalent to the above results and has a number of applications; see [P7,8] and references therein.

13. The von Neumann type minimax theorem

Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be any family of G-convex spaces. Let $X := \prod_{i\in I} X_i$ be equipped with the product topology and $D := \prod_{i\in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D\rangle$, define $\Gamma(A) := \prod_{i\in I} \Gamma_i(\pi_i(A))$. Then (X, D, Γ) is a G-convex space; see [TZ].

Note also that for the case $X_i = D_i$ for each i, the product of G-convex subsets is also G-convex in the product G-space; see [TZ].

In this section, we show that a typical classical application of the KKM theorem can be extended to G-convex spaces.

As a direct application of Theorem 7, we have the following generalization of the von Neumann-Sion minimax theorem:

Theorem 18. *Let $(X;\Gamma)$ and $(Y;\Gamma')$ be compact G-convex spaces and $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(18.1) *$f(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*

(18.2) *for each* $x \in X$, $f(x, \cdot)$ *is l.s.c. and* $g(x, \cdot)$ *is quasiconvex on* Y*; and*

(18.3) *for each* $y \in Y$, $f(\cdot, y)$ *is quasiconcave and* $g(\cdot, y)$ *is u.s.c. on* X.

Then we have

$$\min_{y \in Y} \sup_{x \in X} f(x, y) \leq \max_{x \in X} \inf_{y \in Y} g(x, y).$$

Proof. Note that $y \mapsto \sup_{x \in X} f(x, y)$ is l.s.c. on Y and $x \mapsto \inf_{y \in Y} g(x, y)$ is u.s.c. on X. Therefore, the both sides of the inequality exist. Suppose that there exists a real c such that

$$\max_x \inf_y g(x, y) < c < \min_y \sup_x f(x, y).$$

Let $\Gamma_{X \times Y}$ be the product G-convexity defined as above. Then $(X \times Y; \Gamma_{X \times Y})$ is a compact G-convex space. Define a map $T : X \times Y \multimap X \times Y$ by

$$T(x, y) = \{\overline{x} \in X : f(\overline{x}, y) > c\} \times \{\overline{y} \in Y : g(x, \overline{y}) < c\}$$

for $(x, y) \in X \times Y$. Then $T(x, y)$ is nonempty and Γ-convex for each $(x, y) \in X \times Y$ and $T^{-}(x, y)$ is open. By using Theorem 7, we have an $(x_0, y_0) \in X \times Y$ such that $(x_0, y_0) \in T(x_0, y_0)$. Therefore, $c < f(x_0, y_0) \leq g(x_0, y_0) < c$, a contradiction. □

Remark. If $f = g$ and if X is a convex space, Theorem 18 reduces to Sion's generalization [S] of the von Neumann minimax theorem:

$$\min_x \max_y f(x, y) = \max_y \min_x f(x, y).$$

14. The Nash equilibrium theorem

In this section, from a Fan-Browder type fixed point result for G-convex spaces, we deduce the Ky Fan intersection theorem and the Nash equilibrium theorem for G-convex spaces.

Given a cartesian product $X = \prod_{i=1}^{n} X_i$ of sets, let $X^i = \prod_{j \neq i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$(y_i, x^i) := (x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n).$$

From Corollary 7.2, we have the following Ky Fan type intersection theorem:

Theorem 19. *Let* $X = \prod_{i=1}^n X_i$, $(X;\Gamma)$ *be a compact* G*-convex space, and* $A_1, A_2, \cdots, A_n$ n *subsets of* X *such that*

(19.1) *for each* $x \in X$ *and* $i = 1, \cdots, n$, *the set* $A_i(x) = \{y \in X : (y_i, x^i) \in A_i\}$ *is* Γ*-convex and nonempty; and*

(19.2) *for each* $y \in X$ *and* $i = 1, \cdots, n$, *the set* $A_i(y) = \{x \in X : (y_i, x^i) \in A_i\}$ *is open.*

Then $\bigcap_{i=1}^n A_i \neq \emptyset$.

Proof. Define a map $T : X \multimap X$ by $T(x) = \bigcap_{i=1}^n A_i(x)$ for $x \in X$. Then each $T(x)$ is Γ-convex being an intersection of Γ-convex sets by (19.1). For each $x \in X$ and each i, there exists a $y^{(i)} \in A_i(x)$ by (19.1), or $(y_i^{(i)}, x^i) \in A_i$. Hence, we have $(y_1^{(1)}, \cdots, y_n^{(n)}) \in \bigcap_{i=1}^n A_i(x)$. This shows $T(x) \neq \emptyset$. Moreover, $T^-(y) = \bigcap_{i=1}^n A_i(y)$ is open for each $y \in X$ by (19.2). Now, the conclusion follows from Corollary 7.2. □

Remark. If each X_i is a compact G-convex space, so is X. Note that Theorem 18 can be also deduced from Theorem 19; see [P11].

From Theorem 19, we deduce the following Nash equilibrium theorem for G-convex spaces :

Theorem 20. *Let* $X = \prod_{i=1}^n X_i, (X;\Gamma)$ *a compact* G*-convex space, and* $f_1, \cdots,$
$f_n : X \to \mathbb{R}$ *continuous functions such that*

(20.1) *for each* $x \in X$, *each* $i = 1, \cdots, n$, *and each* $r \in \mathbb{R}$, *the set* $\{(y_i, x^i) \in X : f_i(y_i, x^i) > r\}$ *is* Γ*-convex.*

Then there exists a point $x \in X$ *such that*

$$f_i(x) = \max_{y_i \in X_i} f_i(y_i, x^i) \quad \textit{for} \quad i = 1, \ldots, n.$$

Proof. Let $\varepsilon > 0$ and, for each i, let

$$A_i^\varepsilon = \{x \in X : f_i(x) > \max_{y_i \in X_i} f_i(y_i, x^i) - \varepsilon\}.$$

Then the sets $A_1^\varepsilon, \cdots, A_n^\varepsilon$ satisfy conditions (19.1) and (19.2) of Theorem 19, and hence $\bigcap_{i=1}^n A_i^\varepsilon \neq \emptyset$. Then $H_\varepsilon = \bigcap_{i=1}^n \overline{A_i^\varepsilon}$ is a nonempty compact set. Since $H_{\varepsilon_1} \subset H_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$, we have $\bigcap_{\varepsilon>0} H_\varepsilon \neq \emptyset$. Then $x \in \bigcap_{\varepsilon>0} H_\varepsilon$ satisfies the conclusion. □

Final Remarks. This is not the end of the story. Since there are several hundred published works on the KKM theory, we can cover only an essential part of it. For the more historical background, the reader can consult with [P7,8]. For more involved or generalized versions of the results in this paper, see [P5] for convex spaces, [PK2] for C-spaces, and [P9,10,PK1,3-6] for G-convex spaces and references therein. Note that a number of peoples are currently working on the subject mainly for various equilibrium problems.

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Department of Mathematics
Seoul National University
Seoul 151–742, Korea
e-mail : shpark@math.snu.ac.kr

Bibliographical Sketch and Research Interests

Dr. Sehie Park studied mathematics at Seoul National University (B.S. in 1959 and M.S. in 1961). After having assistantship and lecturership at the University, he became Instructor in 1966 and promoted to Assistant Professor in 1969. In 1972-75, he studied at Indiana University, Bloomington, IN, and received his Ph.D. in 1975. Then he came back to Seoul National University and became Professor in 1981. He taught general and algebraic topology, algebra, differential equations, nonlinear analysis, set theory, history of mathematics, and many subjects in various levels. He had several positions in the Korean Mathematical Society including the Presidentship in 1982-84. He also had a visiting positions at U.C. Berkely, Indiana University, and many foreign institutes. His main research interests include topology, nonlinear analysis, fixed point theory, the KKM theory on general convexities, and others. He published more than 200 papers in journals and proceedings, and more than 180 reviews in MR. He gave almost 200 talks or invited lectures at national or international conferences and various institutes. He is currently a fellow of the Korean Academy of Science and Technology and serving as a member of editorial boards of eight professional scientific journals. He received the Cultural Prize of Seoul City, a national medal for distinguished service, two prizes from the Korean Mathematical Society, and the National Academy of Sciences Prize, Republic of Korea.