

GENERALIZED KKM MAPS ON GENERALIZED CONVEX SPACES

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ABSTRACT. In the present paper, we extend known KKM theorems and matching theorems for generalized KKM maps to G -convex spaces. From these results, we deduce generalized versions of main results of Kassay and Kolumbán [KK] and some others.

1. Introduction

The KKM theory is the study of applications of various equivalent formulations of the classical Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle). It was initiated by the work [KKM] and developed first by Ky Fan [F1,2]. Along with such development, there have appeared numerous generalizations of known results and new applications. For the literature, see Granas [Gr1,2] and Park [P1-3,8-12].

Received March 20, 2000 ; accepted May 20, 2000.

1991 Mathematics Subject Classification: Primary 54H25, 54C60.

Key words and phrases: G -convex space, G -convex, KKM map, generalized KKM, H -convex, H -convex like.

This research was supported by Grant No. 981-0102-013-2 from the Basic Research Program of the Korea Science & Engineering Foundation.

At the beginning, the theory was mainly concerned with convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [L1], and to spaces having certain families of contractible subsets (simply, C -spaces or H -spaces) by Horvath [H1,2]. This line of generalizations of earlier works is followed by Bardaro and Ceppitelli [BC1-3], Ding and Tan [DT], Ding, Kim, and Tan [DKT1,2], Tarafdar [T], Park [P4-6], Park and Kim [PK1], and others.

More recently, generalized notions of KKM maps were introduced by Kassay and Kolumban [KK] and Chang and Zhang [CZ] for convex spaces, and by Chang and Ma [CM] and Kim [K] for H -spaces. On the other hand, in [PK2-6], the authors introduced generalized convex (simply, G -convex) spaces and basic properties of KKM maps for such spaces, which seem to be more adequate for various purposes. Actually, our new concept of G -convex spaces is a common generalization of the usual convexity in a topological vector space and many of abstract convexities which have been developed in connection mainly with the fixed point theory and the KKM theory.

In the present paper, we extend known KKM theorems and matching theorems to G -convex spaces for generalized KKM maps. From these results, we deduce generalized versions of main results of Kassay and Kolumban [KK] and some others.

2. Preliminaries

A *multimap* (or *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y ; that is, a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$.

For any $B \subset Y$, the *lower inverse* and *upper inverse* of B under F are defined by

$$F^{-}(B) = \{x \in X : Fx \cap B \neq \emptyset\} \text{ and } F^{+}(B) = \{x \in X : Fx \subset B\},$$

respectively.

For a set D , let $\langle D \rangle$ denote the set of nonempty finite subsets of D .

Let Δ_n be the standard n -simplex with vertices v_0, v_1, \dots, v_n .

The following is well-known; see [KKM,P11,12]:

THE KKM PRINCIPLE. Let D be the set of vertices of Δ_n and $F : D \multimap \Delta_n$ be a map with closed [resp. open] values such that

$$\text{co } N \subset F(N) \quad \text{for each } N \in \langle D \rangle.$$

Then $\bigcap_{z \in D} Fz \neq \emptyset$.

It is well-known that the closed and the open versions of the KKM principle are equivalent; see [L2].

Let X be a set (in a vector space) and D a nonempty subset of X . Then (X, D) is called a *convex space* [P3] if convex hulls of any $N \in \langle D \rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. A subset A of X is said to be *convex* if, for any $N \in \langle D \rangle$, $N \subset A$ implies $\text{co } N \subset A$, where co denotes the convex hull in the usual sense. If $X = D$, then $X = (X, X)$ reduces to a convex space in the sense of Lassonde [L1]. For a convex space (X, D) , a map $G : D \multimap X$ is called a *KKM map* if $\text{co } N \subset G(N)$ for each $N \in \langle D \rangle$.

A triple $(X, D; F)$ is called an *H-space* [P4,5] if X is a topological space, D a nonempty subset of X , and $F = \{F(A)\}$ a family of contractible subsets of X indexed by $A \in \langle D \rangle$ such that $F(A) \subset F(B)$ whenever $A \subset B \in \langle D \rangle$. If $X = D$, we denote $(X; F)$ instead of $(X, X; F)$ which is called an *H-space* in [BC1-3] and a *C-space* in [H2]. For an $(X, D; F)$ a subset C of X is said to be *H-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $F(A) \subset C$. A map $G : D \multimap X$ is said to be *H-KKM* if $F(A) \subset G(A)$ for each $A \in \langle D \rangle$.

For a set A , let $|A|$ denote the cardinality of A . Let $\mathbb{C}(U, V)$ denote the class of all continuous maps from a topological space U to another V .

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ [P12] consists of a topological space X and a nonempty set D such that for each $N \in \langle D \rangle$ with $|N| = n + 1$, there exist a subset $\Gamma(N)$ of X and a $\phi_N \in \mathbb{C}(\Delta_n, \Gamma(N))$ such that $J \subset N$ implies $\phi_N(\Delta_J) \subset \Gamma(J)$. Note that $\phi_N|_{\Delta_J}$ can be regarded as ϕ_J .

Here Δ_J denotes the face of Δ_n corresponding to $J \in \langle N \rangle$; that is, if $N = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset N$, then $\Delta_J := \text{co}\{v_{i_0}, v_{i_1}, \dots, v_{i_k}\}$. We may write $\Gamma(N) = \Gamma_N$ for each $N \in \langle D \rangle$.

Note that Γ_N does not need to contain N for $N \in \langle D \rangle$. In case $X \supset D$, then $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$. For an $(X \supset D; \Gamma)$, a subset C of X is said to be Γ -convex if for each $N \in \langle D \rangle$, $N \subset C$ implies $\Gamma_N \subset C$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by $(X; \Gamma)$.

It is immediate that convex spaces and H -spaces are particular examples of G -convex spaces. For other example, see [P9-12], [PK2-6].

For a G -convex space $(X, D; \Gamma)$, a map $F : D \multimap X$ is called a *KKM map* if

$$\Gamma_N \subset F(N) \quad \text{for each } N \in \langle D \rangle.$$

From the KKM principle, the first author [P12] deduced the following KKM theorem for G -convex spaces:

THEOREM 1. *Let $(X, D; \Gamma)$ be a G -convex space and $F : D \multimap X$ a map such that*

- (1.1) *F has closed [resp. open] values; and*
- (1.2) *F is a KKM map.*

Then $\{Fz\}_{z \in D}$ has the finite intersection property.

As the KKM principle, it is immediate that the closed and the open versions of the KKM Theorem 1 are mutually equivalent.

From Theorem 1, we obtain the following generalization of the KKM–Fan type theorem [F1]:

COROLLARY. *Under the hypothesis of Theorem 1, if*

- (1.3) *$\bigcap_{z \in M} \overline{Fz}$ is compact for some $M \in \langle D \rangle$, then we have*

$$\bigcap_{z \in M} \overline{Fz} \neq \emptyset.$$

There are lots of more general “compactness” or “coercivity” conditions than (1.3); see [P1-4], [PK1-6].

3. Generalized KKM maps and generalized KKM theorems

Motivated by recent works on generalized KKM maps, we define as follows:

Let I be an index set, which always assumed to be nonempty, and $(X, D; \Gamma)$ a G -convex space. A map $F : I \multimap X$ is called a *generalized KKM map* if for each $J \in \langle I \rangle$, there exists a function $\sigma : J \rightarrow D$ such that for any $M \in \langle J \rangle$, we have $\Gamma_{\sigma(M)} \subset F(M)$. The functional value of F is denoted by F_i or $F(i)$ for $i \in I$.

We give some examples:

(1) Kassay and Kolumbán [KK] and Chang and Zhang [CZ]: Let X be a convex subset of a Hausdorff topological vector space and $\{F_i\}_{i \in I}$ a family of (closed) subsets of X . Suppose that, for each $J \in \langle I \rangle$, there exists a set $\{x_j\}_{j \in J} \subset X$ such that $\text{co}\{x_j\}_{j \in M} \subset \bigcup_{j \in M} F_j$ for each $M \in \langle J \rangle$. Then $F : I \multimap X$ is generalized KKM.

(2) Chang and Ma [CM] and Kim [K]: Let $(X; \Gamma)$ be an H -space and $F : I \multimap X$ be a map. Suppose that, for each $J = \{i_1, \dots, i_n\} \in \langle I \rangle$, there is a $\tilde{J} = \{x_1, \dots, x_n\} \in \langle X \rangle$ such that $\{x_{j_1}, \dots, x_{j_k}\} \in \langle \tilde{J} \rangle$ implies $\Gamma_{\{x_{j_1}, \dots, x_{j_k}\}} \subset \bigcup_{l=1}^k F_{i_{j_l}}$. Define $\sigma : J \rightarrow X$ by $\sigma(i_j) = x_j$ for $j = 1, \dots, n$, then F is generalized KKM for the case $X = D$.

(3) For a G -convex space $(X, D; \Gamma)$, a KKM map $F : D \multimap X$ is a generalized KKM map where σ is the inclusion.

In this section, we extend the known KKM theorems for convex spaces or G -convex spaces for generalized KKM maps.

We begin with the following basic result:

THEOREM 2. *Let $(X, D; \Gamma)$ be a G -convex space, I a nonempty set, and $F : I \multimap X$ a map with closed [resp. open] values.*

(I) *If F is a generalized KKM map, then the family of its values has the finite intersection property.*

(II) *The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$.*

Proof. (I) For each $N \in \langle I \rangle$, there exists a function $\sigma : N \rightarrow D$ such that $M \in \langle N \rangle$ implies $\Gamma_{\sigma(M)} \subset F(M)$. Let $|\sigma(N)| = n + 1$. Then there exists a continuous function $\phi_N : \Delta_n \rightarrow \Gamma_{\sigma(N)}$ such that $\phi_N(\Delta_M) \subset \Gamma_{\sigma(M)}$ for each $M \in \langle N \rangle$, where Δ_M is the face of Δ_n corresponding to $\sigma(M) \subset \sigma(N)$. Since $\Gamma_{\sigma(M)} \subset F(M) \cap \Gamma_{\sigma(N)}$, we have

$$\Delta_M \subset \phi_N^{-1}(\Gamma_{\sigma(M)}) \subset \bigcup \{\phi_N^{-1}(F(i) \cap \Gamma_{\sigma(N)}) : i \in M\}$$

for each $M \in \langle N \rangle$. Note that $F(i) \cap \Gamma_{\sigma(N)}$ is closed [resp. open] in $\Gamma_{\sigma(N)}$ and hence $\phi_N^{-1}(F(i) \cap \Gamma_{\sigma(N)})$ is closed [resp. open] in Δ_n . Moreover, $i \mapsto \phi_N^{-1}(F(i) \cap \Gamma_{\sigma(N)})$ defines a KKM map $F' : N \multimap \Delta_n$ on the G -convex space (Δ_n, N, Γ') , where $\Gamma'_M := \Delta_M$ for each $M \in \langle N \rangle$. Hence, by Theorem 1, we have

$$\bigcap_{i \in N} F'(i) = \bigcap_{i \in N} \phi_N^{-1}(F(i) \cap \Gamma_{\sigma(N)}) \neq \emptyset.$$

This readily implies

$$\Gamma_{\sigma(N)} \cap \bigcap_{i \in N} F(i) \neq \emptyset.$$

Therefore, $\{F(i)\}_{i \in I}$ has the finite intersection property.

(II) Suppose that $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$. For any $N \in \langle I \rangle$, by assumption, we have an $x^* \in \bigcap_{z \in N} F(z) \neq \emptyset$. Define a function $\sigma : N \rightarrow D = X$ by $\sigma(z) = x^*$ for all $z \in N$. Then for any nonempty subset M of N , we have

$$\Gamma_{\sigma(M)} = \Gamma_{\{x^*\}} = \{x^*\} \subset \bigcap_{z \in N} F(z) \subset F(M).$$

Therefore, F is a generalized KKM map.

REMARKS. (1) Note that Theorem 2(I) extends Theorem 1.

(2) For convex spaces, Theorem 2(I) includes Chang and Zhang [CZ, Theorem 3.1], and for a KKM map F , Theorem 2(I) generalizes Knaster, Kuratowski, and Mazurkiewicz [KKM], Dugundji and Granas [DG, Theorem 1.2], Lassonde [L2, Theorem 1], W.K.Kim [Ki, Theorems 1 and 2], and Park [P7, Theorem 6].

(3) For H -spaces $(X; \Gamma)$, Theorem 2(I) extends Chang and Ma [CM, Theorem 1], and H. Kim [K, Theorem 2]. For H -spaces $(X, D; \Gamma)$ and an H -KKM map F , Theorem 2(I) generalizes Park [P4, Theorems 13 and 14].

(4) In recent works of Yuan et al. [KSY, LCY, Y1, Y2], particular forms of Theorem 2 were given with rather lengthy proofs. Moreover, it is trivial that the closed and the open versions of Theorem 2 are

equivalent. Particular cases of this fact were also discussed in the above mentioned works.

From Theorem 2(I), we obtain the following three KKM type theorems for generalized KKM maps with closed values:

THEOREM 3. *Let I be a set, $(X, D; \Gamma)$ a G -convex space, and $F : I \multimap X$ a generalized KKM map with closed values. Suppose that there exists a nonempty compact subset K of X such that either*

- (i) $\bigcap_{i \in J} F_i \subset K$ for some $J \in \langle I \rangle$; or
- (ii) if $X \supset D$ and, for each $J \in \langle I \rangle$ and each function $\sigma : J \rightarrow D$, there exists a compact Γ -convex subset L_N of X containing $N = \sigma(J)$ such that $L_N \cap \bigcap_{i \in J} F_i \subset K$. Then $K \cap \bigcap_{i \in I} F_i \neq \emptyset$.

Proof. Case (i). Note that $\{F_i\}_{i \in I}$ has the finite intersection property by Theorem 2(I). Since $\bigcap_{i \in J} F_i$ is compact, the conclusion easily follows.

Note that, from Case (i), if $X = K$ itself compact, then the conclusion holds without assuming (i) or (ii). From this fact, we can prove Case (ii) as follows:

Case (ii). Suppose that $K \cap \bigcap_{i \in I} F_i = \emptyset$; that is, $K \subset \bigcup_{i \in I} (X \setminus F_i)$. Since each $X \setminus F_i$ is open and K is compact, there exists a $J \in \langle I \rangle$ such that $K \subset \bigcup_{i \in J} (X \setminus F_i)$. Since $F : I \multimap X$ is generalized KKM, there exists a function $\sigma : J \rightarrow D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle J \rangle$. Let $N = \sigma(J) \in \langle D \rangle$ and L_N be the set in (ii). Define $F' : J \multimap L_N$ by $F'_j = F_j \cap L_N$ for each $j \in J$. Then each F'_j is closed in L_N . For each $M \in \langle J \rangle$, since $\Gamma_{\sigma(M)} \subset F(M)$, we have

$$\Gamma_{\sigma(M)} \subset F(M) \cap L_N \subset F'(M).$$

Therefore, F' is generalized KKM and L_N is compact, we have $\bigcap_{j \in J} F'_j = L_N \cap \bigcap_{j \in J} F_j \neq \emptyset$. Let $z \in L_N \cap \bigcap_{j \in J} F_j$. If $z \in K$, then

$$z \in K \subset \bigcup_{j \in J} (X \setminus F_j)$$

and hence $z \notin F_j$ for some $j \in J$, which is a contradiction. Therefore, we have $z \in L_N \setminus K$. This implies $z \notin \bigcap_{j \in J} F_j$ by (ii), which leads another contradiction. Therefore, we must have $K \cap \bigcap_{i \in I} F_i \neq \emptyset$. This completes our proof.

REMARK. If $(X; \Gamma)$ is an H -space, Theorems 2(I) and 3 reduce to Kim [K, Theorem 1] and improve Chang and Ma [CM, Theorem 1]. If $(X, D; \Gamma)$ is an H -space and $I = D$, Theorem 3 extends Park [P5, Theorem 1]. Further, if $X = D$ is a convex subset of a topological vector space, Theorem 3 improves Chang and Zhang [CZ, Theorem 3.1] and Kassay and Kolumban [KK, Theorem 3.1].

From now on, in this section, we consider the case $I = D$ for the simplicity:

THEOREM 4. *Let $(X \supset D; \Gamma)$ be a G -convex space, and $F : D \multimap X$ a map with closed values. For a compact Γ -convex subset L of X , if the multimap $F' : D \cap L \multimap L$ defined by $F'x = Fx \cap L$ is a generalized KKM map, then we have*

$$L \cap \bigcap \{Fx : x \in L \cap D\} \neq \emptyset.$$

Proof. Consider $(L, L \cap D, F')$ instead of (X, D, F) in Theorem 3 with $I = D$. Then all of the requirements are satisfied. Therefore,

$$\bigcap \{F'x : x \in L \cap D\} = L \cap \bigcap \{Fx : x \in L \cap D\} \neq \emptyset.$$

This completes our proof.

REMARK. Park [P4, Theorem 2] is a particular case of Theorem 4 for an H -KKM map F .

THEOREM 5. *Let $(X \supset D; \Gamma)$ be a G -convex space, Y a topological space, $F : D \multimap Y$, $G : X \multimap Y$ multimaps, and K a nonempty compact subset of Y . Suppose that*

- (5.1) *for each $x \in D$, Fx and $G^+(Fx)$ are closed in Y ;*
- (5.2) *for each compact Γ -convex subset L of X , $(G^+F)' : D \cap L \multimap L$ defined by $(G^+F)'x = G^+(Fx) \cap L$ is a generalized KKM map; and*
- (5.3) *for each $N \in \langle D \rangle$, we have a compact Γ -convex subset L_N of X containing N such that*

$$L_N \cap \bigcap \{(G^+F)x : x \in L_N \cap D\} \subset G^+(K).$$

Then we have

$$\overline{G(X)} \cap K \cap \bigcap \{Fx : x \in D\} \neq \emptyset.$$

Proof. Suppose that $\overline{G(X)} \cap K \cap \bigcap \{Fx : x \in D\} = \emptyset$. Since $\overline{G(X)} \cap K$ is compact and contained in $\bigcup_{x \in D} (Y \setminus Fx)$, by (5.1), there exists an $N \in \langle D \rangle$ such that

$$\overline{G(X)} \cap K \subset \bigcup_{x \in N} (Y \setminus Fx).$$

Since we have an $L_N \subset X$ as in (5.3),

$$L_N \cap \bigcap_{x \in L_N \cap D} G^+Fx \cap G^+(K) = \emptyset.$$

However, by (5.3), we have

$$L_N \cap \bigcap_{x \in L_N \cap D} G^+Fx \subset G^+(K).$$

Therefore, we have

$$L_N \cap \bigcap_{x \in L_N \cap D} G^+Fx = \emptyset.$$

This contradicts Theorem 4.

REMARK. For an H -space $(X; \Gamma)$ and $G \in \mathbb{C}(X, Y)$, Theorem 5 improves Chang and Ma [CM, Theorem 3]. For an H -space $Y = (X \supset D; \Gamma)$ and $F = 1_X$, Theorem 5 extends Park [P5, Theorem 1(ii)].

4. Matching theorems

From Theorem 5 we obtain the following matching theorems for open covers:

THEOREM 6. *Let $(X \supset D; \Gamma)$ be a G -convex space, Y a topological space, $F : D \rightarrow Y$, $G : X \rightarrow Y$ maps, and K a nonempty compact subset of Y . Suppose that*

- (6.1) *for each $x \in D$, Fx and $G^-(Fx)$ are open in Y ;*
- (6.2) *$\overline{G(X)} \cap K \subset F(D)$; and*
- (6.3) *for each $N \in \langle D \rangle$, we have a compact Γ -convex subset L_N of X containing N such that $L_N \setminus G^+(K) \subset \bigcup \{G^-(Fx) : x \in L_N \cap D\}$. Then there exist an $I \in \langle D \rangle$ and an $x_0 \in \Gamma_I$ satisfying $Gx_0 \cap Fx \neq \emptyset$ for each $x \in I$.*

Proof. Let $Sx = Y \setminus Fx$ for $x \in D$. Then Sx is closed in Y for each $x \in D$. Suppose that the conclusion is false. Then for any $I \in \langle D \rangle$ and $z \in \Gamma_I$, we have $Gz \cap Fx = \emptyset$ for some $x \in I$. So $Gz \subset S(I)$; that is,

$$\Gamma_I \subset G^+S(I).$$

Therefore, G^+S is generalized KKM. Since (6.3) implies (5.3), all of the requirements of Theorem 5 are satisfied. So we have $\overline{G(X)} \cap K \cap \bigcap \{Sx : x \in D\} \neq \emptyset$. This contradicts (6.2). This completes our proof.

REMARK. Note that if $G = g \in \mathbb{C}(X, Y)$ the conclusion of Theorem 6 implies that there exists an $A \in \langle D \rangle$ and an $x_0 \in \Gamma_A$ such that $gx_0 \in \bigcap_{x \in A} Fx$. Therefore, Theorem 6 improves Park [P4, Theorem 5].

THEOREM 7. *Under the hypothesis of Theorem 6 without condition (6.3), if X is compact, then there exists an $I \in \langle D \rangle$ such that, for each $\sigma : I \rightarrow D$, there exist a $J \in \langle I \rangle$ and an $x_0 \in \Gamma_{\sigma(J)}$ satisfying $Gx_0 \cap Fx \neq \emptyset$ for each $x \in J$.*

Proof. Let $Sx = Y \setminus Fx$ for $x \in D$. Then Sx is closed in Y for each $x \in D$. Suppose that the conclusion is false. Then for any $I \in \langle D \rangle$ there exists a $\sigma : I \rightarrow D$ such that, for each $J \in \langle I \rangle$ and $z \in \Gamma_{\sigma(J)}$, we have $Gz \cap Fx = \emptyset$, for some $x \in J$. So $Gz \subset S(J)$; that is,

$$\Gamma_{\sigma(J)} \subset G^+S(J).$$

Therefore, G^+S is generalized KKM. Since X is compact, all of the requirements of Theorem 5 are satisfied. So we have $\overline{G(X)} \cap K \cap \bigcap \{Sx : x \in D\} \neq \emptyset$. This contradicts (6.2). This completes our proof.

REMARK. Theorem 7 extends Chang and Ma [CM, Theorem 2 and Corollary 1].

The following simple consequence of Theorem 2(I) is a result for generalized KKM maps with open values:

THEOREM 8. *Let $(X \supset D; \Gamma)$ be a G -convex space, Y a topological space, $F : D \multimap Y$, and $G : X \multimap Y$. Suppose that*

- (8.1) *for each $x \in D$, G^+Fx is open in Y ; and*
 (8.2) *$G^+F : D \rightarrow X$ is a generalized KKM map. Then (i) $\{Fx : x \in D\}$ has the finite intersection property; and (ii) for any Γ -convex subset L of X such that $L \cap D$ is finite, we have*

$$G(L) \cap \bigcap \{Fx : x \in L \cap D\} \neq \emptyset.$$

Proof. (i) Since G^+F is a generalized KKM map with open values, for any $J \in \langle D \rangle$, by Theorem 2(I), we have

$$G^+ \bigcap_{x \in J} (Fx) = \bigcap_{x \in J} G^+Fx \neq \emptyset,$$

whence we have $\bigcap_{x \in J} Fx \neq \emptyset$.

(ii) Define $F'x = Fx \cap G(L)$ for $x \in L \cap D$. Then $F' : L \cap D \multimap Y$ is well-defined. Consider $(L \cap D, L, G(L), F')$ instead of (D, X, Y, F) in (i). Then all of the requirements are satisfied. Therefore, by (i), we have

$$\bigcap \{F'x : x \in L \cap D\} = G(L) \cap \bigcap \{Fx : x \in L \cap D\} \neq \emptyset.$$

REMARK. For H -spaces, Theorem 8(i) improves Chang and Ma [CM, Theorem 4] and Park [P4, Theorem 14]. For convex spaces, Theorem 8(ii) reduces to Park [P5, Corollary 4] and Lassonde [L2, Theorem 1].

From Theorem 8(i), we obtain the following matching theorem for closed covers:

THEOREM 9. *Let $(X \supset D; \Gamma)$ be a G -convex space, Y a topological space, and $t \in \mathbb{C}(X, Y)$. Let C_1, C_2, \dots, C_n be n closed subsets of Y such that $Y = \bigcup_{i=1}^n C_i$. Then, for any $I = \{x_1, x_2, \dots, x_n\} \in \langle D \rangle$, there exists a $J \in \langle I \rangle$ such that $t(\Gamma_J) \cap \bigcap \{C_i : x_i \in J\} \neq \emptyset$.*

Proof. Suppose that there exists an $I = \{x_1, x_2, \dots, x_n\} \in \langle D \rangle$, where x_i 's are not necessarily distinct, such that, for any $J \in \langle I \rangle$, $t(\Gamma_J) \cap \bigcap \{C_i : x_i \in J\} = \emptyset$; that is,

$$t(\Gamma_J) \subset Y \setminus \bigcap \{C_i : x_i \in J\} = \bigcup \{Y \setminus C_i : x_i \in J\}.$$

For $D = I$ and $Fx_i = Y \setminus C_i$ for $x_i \in I$, F satisfies all of the requirements of Theorem 8(i). Therefore, we have

$$Y \setminus \bigcup \{C_i : x_i \in I\} = \bigcap \{Fx_i : x_i \in I\} \neq \emptyset.$$

This contradicts $Y = \bigcup_{i=1}^n C_i$.

REMARK. For H -spaces, Theorem 9 generalizes Park [P4, Theorem 13] and Chang and Ma [CM, Corollary 3].

Finally, all of the results of Chang and Ma [CM] not mentioned in this section are consequences of Park [P4,5] and Park and Kim [PK1].

5. Variations of KKM theorems and matching theorems

In this section, we deduce generalized versions of main results of Kassay and Kolumbán [KK] from our results in Section 3. Actually, in Section 3 of [KK], the authors obtained two KKM type theorems [KK, Theorem 3.1 and 3.5] and two matching theorems [KK, Theorems 3.2 and 3.4] on open covers of convex sets. We generalize those results to G -convex spaces under more general situations. In fact, our Theorems 10 and 11 contain all those four theorems in Section 3 of [KK] as particular cases.

The following is the matching theorem on open covers, which follows from Theorem 6.

THEOREM 10. *Let $(X; \Gamma)$ be a G -convex space, K a nonempty compact subset of X , I an index set, $I_0 \subset I$, and $\varphi : I \rightarrow X$. Let $A : I \multimap X$ be a map such that*

- (10.1) *for each $i \in I$, A_i is open in X ;*
- (10.2) *$K \subset A(I)$; and*
- (10.3) *for each $N \in \langle X \rangle$, there exists a compact Γ -convex subset L_N of X containing N such that $\varphi(I_0) \subset L_N$ and $L_N \setminus K \subset A(I_0)$.*
Then there exists a $J \in \langle I \rangle$ such that

$$\Gamma_{\varphi(J)} \cap \bigcap_{j \in J} A_j \neq \emptyset.$$

Proof. In Theorem 6, let $X = Y$, $F = 1_X$, and $D = \varphi(I)$. Let $F : D \multimap X$ be defined by $Fx = \bigcup \{A_i : x = \varphi(i), i \in I\}$ for each $x \in D$. Then

- (6.1) Fx is open in X for each $x \in D$;
- (6.2) $\overline{X} \cap K = K \subset A(I) \subset F(D)$ since, for each $i \in I$, there exists an $x = \varphi(i) \in D$ such that $A_i \subset Fx$; and
- (6.3) $L_N \setminus K \subset A(I_0) \subset F(L_N \cap D)$ since, for each $i_0 \in I_0$, there exists an $x = \varphi(i_0) \in L_N \cap D$ and $A_{i_0} \subset Fx$.

Therefore, by Theorem 6, there exists an $J' \in \langle D \rangle$ and an $x_0 \in \Gamma_{J'}$ satisfying $x_0 \in \bigcap_{x \in J'} Fx$. For each $x \in J'$, choose a $j_x \in I$ such that $j_x \in \varphi^{-1}x$ and $x_0 \in A_{j_x} \subset Fx$. Put $J = \{j_x : x \in J'\}$, then $J' = \varphi(J)$, $J \in \langle I \rangle$ and

$$x_0 \in \Gamma_{\varphi(J)} \cap \bigcap_{j \in J} A_j.$$

This completes our proof.

REMARK. Even for a convex space X , Theorem 10 generalizes Kasay and Kolumbán [KK, Theorem 3.4] which extends Fan [F2, Theorem 3].

From Theorem 10, we obtain the following:

COROLLARY. Let $(X; \Gamma)$ be a G -convex space, L a compact Γ -convex subset of $(X; \Gamma)$, I a set, $A : I \multimap X$ a map, and $\varphi : I \rightarrow L$ a function. Suppose that

- (1) for each $i \in I$, A_i is open in X ; and
- (2) $L \subset A(I)$.

Then there exists a $J \in \langle I \rangle$ such that

$$\Gamma_{\varphi(J)} \cap L \cap \bigcap_{j \in J} A_j \neq \emptyset.$$

Proof. Let $D = \varphi(I)$. Then $(L, D; \Gamma)$ is a compact G -convex space. Let $K = L$ and define $A' : I \multimap L$ by $A'_i = A_i \cap L$ for $i \in I$. Then

- (10.1) for each $i \in I$, A'_i is open in L ;
- (10.2) $K = L \subset A(I) \cap L = A'(I)$; and
- (10.3) put $X = L = K$ and $I_0 = I$.

Therefore, by Theorem 10, the conclusion follows.

REMARK. For a convex space X , Corollary reduces to Kassay and Kolumbán [KK, Theorem 3.2], which extends Fan [F2, Lemma 1] whenever φ is injective.

Note that Theorem 10 is equivalent to the following KKM type theorem:

THEOREM 11. Let $(X; \Gamma)$ be a G -convex space, K a nonempty compact subset of X , I a set, $I_0 \subset I$, and $\varphi : I \rightarrow X$. Let $B : I \multimap X$ be a map such that

- (11.1) for each $i \in I$, B_i is closed in X ;
- (11.2) for each $J \in \langle I \rangle$, $\Gamma_{\varphi(J)} \subset B(J)$; and
- (11.3) for each $N \in \langle X \rangle$, there exists a compact Γ -convex subset L_N of X containing N such that $\varphi(I_0) \subset L_N$ and

$$L_N \cap \bigcap_{i \in I_0} B_i \subset K.$$

Then we have

$$K \cap \bigcap_{i \in I} B_i \neq \emptyset.$$

Proof. Suppose that the conclusion is false. For each $i \in I$, let $A_i = X \setminus B_i$. Then

(10.1) for each $i \in I$, A_i is open in X ;

(10.2) $K \subset (\bigcap_{i \in I} B_i)^c = \bigcup_{i \in I} (X \setminus B_i) = A(I)$; and

(10.3) $L_N \setminus K \subset (\bigcap_{i \in I_0} B_i)^c = \bigcup_{i \in I_0} (X \setminus B_i) = A(I_0)$.

Therefore, by Theorem 10, there exists a $J \in \langle I \rangle$ such that $\Gamma_{\varphi(J)} \cap \bigcap_{j \in J} A_j \neq \emptyset$. This contradicts (11.2).

REMARK. Conversely, we can obtain Theorem 10 from Theorem 11. Even for a convex space X , Theorem 11 generalizes Kassay and Kolumbán [KK, Theorem 3.5], which extends Fan [F2, Theorem 4]. From Theorem 11, we can also obtain [KK, Theorem 3.1], which is already shown to be a consequence of Theorem 3.

6. Applications

In [KK], from a particular form of Theorem 11, the authors deduced some applications to properties of convex sets, fixed points, variational and minimax inequalities. In this section, we extend key results of Section 4 of [KK] to G -convex spaces.

We need to define some new notions as follows:

A pair $(X; \Gamma)$ is called an H -set if X is a nonempty set and $\Gamma = \{\Gamma_A\}$ a family of subsets of X indexed by $A \in \langle X \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle X \rangle$. A subset C of X is said to be H -convex if $A \in \langle C \rangle$ implies $\Gamma_A \subset C$. A convex subset of a vector space is always an H -set by putting $\Gamma_A = \text{co}A$ and so is an H -space $(X; \Gamma)$.

For H -sets $(X; \Gamma)$ and $(Y; \Gamma')$, a function $\varphi : X \rightarrow Y$ is said to be H -convex-like if, for any $J \in \langle X \rangle$, we have $\Gamma'_{\varphi(J)} \subset \varphi(\Gamma_J)$. An H -convex-like function between convex sets is said to be *convex-like* as in [KK].

From Theorem 11, we deduce the following section property of H -sets:

THEOREM 12. *Let $(X; \Gamma)$ be an H -set, $(Y; \Gamma')$ a G -convex space, K a nonempty compact subset of Y , Z a set, $h : Y \rightarrow Z$ a function, and $A \subset B \subset X \times Z$. Suppose that*

- (12.1) *for each $x \in X$, $\{y \in Y : (x, hy) \in B\}$ is closed in Y ;*
- (12.2) *for each $y \in Y$, $\{x \in X : (x, hy) \notin A\}$ is H -convex or empty;*
- (12.3) *there is an H -convex-like function $\varphi : X \rightarrow Y$ such that $(x, (h\varphi)x) \in A$ for all $x \in X$; and*
- (12.4) *for each $N \in \langle Y \rangle$, there exists a compact Γ' -convex subset L_N of Y containing N such that, for any subset X_0 of X satisfying $\varphi(X_0) \subset L_N$, we have*

$$L_N \cap \bigcap_{x \in X_0} \{y \in Y : (x, hy) \in B\} \subset K.$$

Then there exists a $y_0 \in K$ such that $X \times \{hy_0\} \subset B$.

Proof. For each $x \in X$, let $Fx = \{y \in Y : (x, hy) \in B\}$. Then each Fx is closed by (12.1). Let $J \in \langle X \rangle$ and $y \in \Gamma'_{\varphi(J)}$. Then $y \in \varphi(\Gamma_J) \subset Y$ since φ is H -convex-like. Hence $y = \varphi(\bar{x})$ for some $\bar{x} \in \Gamma_J$. Suppose that $y \notin F(J)$. Then $y \notin Fx$ for each $x \in J$; that is, $(x, hy) \notin B$ and hence, $(x, hy) \notin A$ for each $x \in J$. By (12.2), for each $\bar{x} \in \Gamma_J$, we have $(\bar{x}, hy) \notin A$; that is, $(\bar{x}, (h\varphi)\bar{x}) \notin A$. This contradicts (12.3).

Hence $\Gamma'_{\varphi(J)} \subset F(J)$. Now, by applying Theorem 11 with (X, Y, F, X_0) instead of (I, X, B, I_0) , there exists a point $y_0 \in K \cap \bigcap_{x \in X} Fx$; that is, $y_0 \in K$ and $X \times \{hy_0\} \subset B$. This completes our proof.

REMARK. Even for convex spaces, Theorem 12 generalizes [KK, Theorem 4.4], which extends Lin [Li, Theorem 1]. By putting $Y = K = Z$ and $h = 1_Y$ in Theorem 12, we obtain a generalization of [KK, Corollary 4.3].

From Theorem 12, we obtain the following analytic alternative and minimax inequality:

THEOREM 13. *Let (X, Γ) be an H -set, $(Y; \Gamma')$ a G -convex space, K a nonempty compact subset of Y , $\varphi : X \rightarrow Y$ an H -convex-like function, and $f, g : X \times Y \rightarrow \overline{\mathbb{R}}$. Suppose that, for any $\alpha \leq \beta$ in $\overline{\mathbb{R}}$, the followings hold:*

- (13.1) $g(x, y) \leq f(x, y)$ for all $(x, y) \in X \times Y$;
- (13.2) for each $x \in X$, $\{y \in Y : g(x, y) > \beta\}$ is open;
- (13.3) for each $y \in Y$, $\{x \in X : f(x, y) > \alpha\}$ is H -convex or empty; and
- (13.4) for each $N \in \langle Y \rangle$, there exists a compact Γ' -convex subset L_N of Y containing N such that, for any subset X_0 of X satisfying $\varphi(X_0) \subset L_N$, we have

$$\{y \in L_N : g(x, y) \leq \beta \text{ for all } x \in X_0\} \subset K.$$

(I) *Then either*

- (i) *there exists a $\hat{y} \in K$ such that $g(x, \hat{y}) \leq \beta$ for all $x \in X$, or*
- (ii) *there exists an $\hat{x} \in X$ such that $f(\hat{x}, \varphi(\hat{x})) > \alpha$.*

(II) *Further if $\alpha = \beta = \sup_{x \in X} f(x, \varphi(x))$, then we have*

$$\min_{y \in K} \sup_{x \in X} g(x, y) \leq \sup_{x \in X} f(x, \varphi(x)).$$

Proof. Suppose that (ii) does not hold; that is, $f(x, \varphi(x)) \leq \alpha$ for all $x \in X$. Let

$$A = \{(x, y) \in X \times Y : f(x, y) \leq \alpha\},$$

$$B = \{(x, y) \in X \times Y : g(x, y) \leq \beta\}.$$

Then $A \subset B$ by (13.1) and $\alpha \leq \beta$. Now, apply Theorem 12 with $Y = Z$ and $h = 1_Y$. Then (13.2)-(13.4) imply (12.1)-(12.3). Therefore, there exists a $\hat{y} \in K$ such that $X \times \{\hat{y}\} \subset B$. This implies (i). Note that (II) clearly follows from (I).

REMARK. Theorem 13 generalizes [KK, Theorems 4.4, 4.9, and 4.10]. The authors applied [KK, Theorem 4.4] to obtain generalizations of Browder's variational inequality [Br, Theorem 2] and Fan's best approximation theorem [F2, Theorem 7].

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