COMMENTS ON A COINCIDENCE THEOREM IN GENERALIZED CONVEX SPACES

BY

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Abstract. We give a corrected version of the KKM theorem in [2] and a coincidence theorem for two set-valued maps. Moreover, a counter-example of a fixed point result for compact G-convex spaces in [2] is given.

In 1993, the first author [17] introduced the concept of generalized convex spaces (or G-convex spaces) as a far-reaching generalization of various general convexities without linear structure due to a large number of other authors. We established in such a context the foundations of the KKM theory initiated by Knaster, Kuratowski, and Mazurkiewicz [9], as well as fixed point theorems and other results for multimaps; see [11-20, 7]. This direction of study was immediately followed by Tan et al. [2, 22, 23] and other authors.

Especially, in [2], a coincidence theorem for set-valued maps from a compact G-convex space to a uniform space is obtained, generalizing Ky Fan's work [4] on the Tychonoff fixed point theorem to set-valued maps on G-convex spaces. However, unfortunately, there are in [2] some incorrect statements.

The main aim in this paper, we clarify this matter and give some comments. In fact, Theorem 1 is a G-convex space version of the KKM theorem of Ky Fan [4] and Theorem 2 is an abstract version of the coincidence theorems in [2]. Finally, we show the invalidity of [2, Corollary 2.5] by giving a counter-example.

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A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ having $n+1$ elements, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Note that $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $e_0, e_1, \ldots, e_n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by (X, Γ).

In our previous works [11-20], we assumed that D is a nonempty subset of X in the definition of a G-convex space $(X, D; \Gamma)$. From now on, we adopt the above definition; that is, we do not assume $D \subset X$.

There are a large number of examples of G-convex spaces; see [1, 5, 6, 16-18].

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

The following is well-known; see [8-11, 21]:

The KKM Principle. *Let D be the set of vertices of Δ_n and $F : D \multimap \Delta_n$ be a KKM map* (*that is,* $\mathrm{co}\, A \subset F(A)$ *for each* $A \in \langle D \rangle$) *with closed* [*resp. open*] *values. Then $\{F(a)\}_{a \in D}$ has nonempty intersection.*

The following is a KKM theorem for G-convex spaces:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap with nonempty compactly closed* [*resp. open*] *values. Suppose that F is a KKM map. Then*

(i) *$\{F(a)\}_{a \in D}$ has the finite intersection property; and*

(ii) *if F has compactly closed values and if $\bigcap_{a \in M} F(a)$ is compact for some $M \in \langle D \rangle$, then we have $\bigcap_{a \in D} F(a) \neq \emptyset$.*

Proof. Let $A = \{a_0, a_1, \dots, a_n\} \in \langle D \rangle$. Then there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A$ such that, for any $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, we have

$$\phi_A(\mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \dots, a_{i_k}\}) \cap \phi_A(\Delta_n).$$

Since F is a KKM map, it follows that

$$\begin{aligned} \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\} &\subset \phi_A^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \dots, a_{i_k}\}) \cap \phi_A(\Delta_n)) \\ &\subset \bigcup_{j=0}^{k} \phi_A^{-1}(F(a_{i_j}) \cap \phi_A(\Delta_n)). \end{aligned}$$

Since $F(a_{i_j}) \cap \phi_A(\Delta_n)$ is closed [resp. open] in the compact subset $\phi_A(\Delta_n)$ of Γ_A, $\phi_A^{-1}(F(a_{i_j}) \cap \phi_A(\Delta_n))$ is closed [resp. open] in Δ_n. Note that $e_i \multimap \phi_A^{-1}(F(a_i) \cap \phi_A(\Delta_n))$ is a KKM map on $\{e_0, e_1, \dots, e_n\}$. Hence, by the KKM principle, we have

$$\bigcap_{i=0}^{n} \phi_A^{-1}(F(a_i) \cap \phi_A(\Delta_n)) \neq \emptyset,$$

which readily implies $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$. This completes the proof of (i).

Note that (ii) follows immediately from (i).

Remarks. (a) Theorem 1 is an improved version of [2, Lemma 1.4], where X should be a subset of (Y, Γ) and Y is not necessarily completely regular nor Hausdorff.

(b) A far-reaching generalized form of Theorem 1 is already given by the first author and Kim [18] with a different method.

(c) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \mathrm{co}$, the convex hull, Theorem 1 reduces to the celebrated KKM principle [9].

(d) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 1 extends Fan's KKM theorem [4].

Recall that a set-valued map is said to be *compact* if its range is relatively compact.

The following is a coincidence theorem for set-valued maps:

Theorem 2. *Let X be a topological space, Z a Hausdorff uniform space, and $S, T : X \multimap Z$ compact upper semicontinuous maps with nonempty closed values such that*

(1) *for any entourage V of Z,*

$$\Phi(V) := \{x \in X : (S(x) \times T(x)) \cap \overline{V} \neq \emptyset\} \neq \emptyset;$$

(2) *for some entourage V_0 of Z, $\Phi(V_0)$ is compact, where $\overline{V}$ is the closure of V in $Z \times Z$.*

Then S and T have a coincidence point $\hat{x} \in X$; that is, $S(\hat{x}) \cap T(\hat{x}) \neq \emptyset$.

Proof. Let $\mathcal{U}$ be the basis of open entourages of the uniformity of Z. Since S and T are compact upper semicontinuous maps with closed values, by [3, Lemma 3], so is $S \times T$. Therefore, $\Phi(V)$ is closed in X for each $V \in \mathcal{U}$.

For any finite subset $\{V_1, V_2, \dots, V_n\}$ of $\mathcal{U}$, we have

$$\bigcap_{i=1}^{n} V_i \in \mathcal{U} \quad \text{and} \quad \bigcap_{i=1}^{n} \Phi(V_i) \supset \Phi(\bigcap_{i=1}^{n} V_i).$$

Hence, $\{\Phi(V) : V \in \mathcal{U}\}$ is a family of nonempty closed subsets of X with the finite intersection property. Since $\Phi(V_0)$ is compact, we have an $\hat{x} \in \bigcap_{V \in \mathcal{U}} \Phi(V)$. Then

$$(S(\hat{x}) \times T(\hat{x})) \cap \overline{V} \neq \emptyset \quad \text{for all} \quad V \in \mathcal{U}.$$

By [CT, Lemma 2.2], we have $S(\hat{x}) \cap T(\hat{x}) \neq \emptyset$.

Remarks. (a) We modified the proof of [2, Theorem 2.3], which is a consequence of Theorem 2 for the case $X = (X, \Gamma)$ is a compact G-convex space. Its authors assumed some Ky Fan type conditions implying condition (1).

(b) Moreover, it was claimed that the following [2, Corollary 2.5] can be deduced from [2, Theorem 2.3]:

Statement. *Let (X, Γ) be a compact Hausdorff G-convex space and $S : X \to X$ be continuous. Then S has a fixed point in X.*

This is a rather surprising result since it readily implies the validity of the Schauder conjecture: every compact convex subset of a Hausdorff topological vector space has the fixed point property. This conjecture is not resolved yet even when the space is metrizable.

The following example shows the invalidity of the statement:

Counter-Example. Let $X = D = [0, 1)$ and $Y = D' = \mathbb{S}^1 = \{e^{2\pi it} : t \in [0, 1)\}$ in the complex plane $\mathbb{C}$. Let $f : X \to Y$ be a continuous bijection defined by $f(t) = e^{2\pi it}$. Define $\Gamma : \langle D' \rangle \multimap Y$ by

$$\Gamma(A) = f(\mathrm{co}(f^{-1}(A))) \quad \text{for} \quad A \in \langle D' \rangle.$$

Then $(Y, D'; \Gamma)$ is a compact G-convex space. (More generally, it is shown in [16] that any continuous image of a G-convex space is a G-convex space.) Note that $\mathbb{S}^1$ lacks the fixed point property.

Moreover, this is an example of a G-convex space $(Y, D; \Gamma)$ satisfying $D \not\subset Y$. In fact, let us define $\Gamma : \langle D \rangle \multimap Y$ by

$$\Gamma(A) = f(\mathrm{co}\, A) \quad \text{for} \quad A \in \langle D \rangle.$$

Therefore, our new G-convex spaces are general than old ones.

In view of Theorem 2, we obtain the following corrected form of the statement:

Corollary 3. *Let X be a compact Hausdorff uniform space and $f : X \to X$ a continuous map. Then f has a fixed point if and only if*

$(1)'$ *for any $V \in \mathcal{U}$, $\mathrm{Gr}(f) \cap \overline{V} \neq \emptyset$,*

where $\mathcal{U}$ is the basis of the uniformity of X.

Proof. Note that $(1)'$ is equivalent to (1) for the case $S = f$ and $T = 1_X$.

It is interesting to notice the following:

Corollary 4. *Let X be a compact subset of a Hausdorff topological vector space E, and $f : X \to X$ a continuous map. Then f has a fixed point if and only if*

(1)$''$ *for any neighborhood V of the origin 0 of E, there exists an $x \in X$ such that $x - f(x) \in V$.*

Finally, for historical background on analytical fixed point theory and the KKM theory, see [11].

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