REMARKS ON A FIXED POINT PROBLEM OF BEN-EL-MECHAIEKH

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A *multimap* or *map* $T : X \multimap Y$ is a function from X to the power set of Y. Let $x \in T^{-}y$ if and only if $y \in Tx$.

A *polytope* is a convex hull of a nonempty finite number of elements in a topological vector space or a compact convex subset of a finite dimensional subspace.

For any topological spaces X, Y and given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite compositions of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following properties:

- (i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
- (ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous and nonempty compact-valued; and
- (iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

We introduce two more classes:

$F \in \mathfrak{A}_c^{\pi}(X, Y) \iff$ for any paracompact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^{\kappa}(X, Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^{\pi} \subset \mathfrak{A}_c^{\kappa}$. Any class $\mathfrak{A}_c^{\kappa}$ will be called *admissible*. For details, see [P1-4, PK1-3].

In [PK1-3], we introduced the following unified generalization of various general convexities without any linear structure:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X, and a multimap $\Gamma : \langle D \rangle \multimap X$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D, such that for each $A \in \langle D \rangle$ having $n + 1$ elements, there

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exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where Δ_J denotes the face of the standard n-simplex Δ_n corresponding to $J \in \langle A \rangle$.

We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$. In $(X, D; \Gamma)$, a subset C of X is called a Γ-*convex set* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. If $X = D$, then we denote $(X, \Gamma) = (X, X; \Gamma)$.

Major examples of G-convex spaces are convex subsets of topological vector spaces, Lassonde's convex spaces, C-spaces (or H-spaces) due to Horvath, metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joo's pseudoconvex spaces, and so on. For the literature, see [PK1-3].

Recently, in [P9], we give some new subclasses of the class of G-convex spaces and, similtaneously, show that some recent abstract convexities of other authors are simple particular examples of our G-convexity. Such subclasses are L-spaces of Ben-El-Mechaiekh *et al.*, continuous images of C-spaces, Verma's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, generalized H-spaces of Stachó, and Llinares' mc-spaces.

We need the following coincidence theorem:

Theorem 1. [PK2,3, Theorem 1] *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y, T : X \multimap Y$ two maps, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1.1) *for each $x \in D$, Sx is compactly open in Y;*
(1.2) *for each $y \in F(X)$, $M \in \langle S^- y \rangle$ implies $\Gamma_M \subset T^- y$;*
(1.3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*
(1.4) *either*
 (i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D \rangle$; or*
 (ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X containing N such that $F(L_N) \backslash K \subset S(L_N \cap D)$.*

Then there exists an $\overline{x} \in X$ such that $F\overline{x} \cap T\overline{x} \neq \emptyset$.

Note that if F is single-valued, we do not need the Hausdorffness of Y, and that if F is a compact map then (1.4)(ii) holds automatically.

For $X = Y = K$ and $F = 1_X$, by replacing S^- and T^- by S and T, respectively, Theorem 1 reduces to the following:

Theorem 2. *Let $(X, D; \Gamma)$ be a compact G-convex space, and $S : X \multimap D$, $T : X \multimap X$ maps such that*

(2.1) *for each $x \in X$, $M \in \langle Sx \rangle$ implies $\Gamma_M \subset Tx$; and*
(2.2) *$X = \bigcup \{\text{Int}\, S^- y : y \in D\}$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in Tx_0$.

If X is a convex subset of a topological vector space and $\Gamma_N = \text{co}\, N$ for $N \in \langle D \rangle$, then Theorem 2 is a generalization of the Fan-Browder fixed point theorem and has numerous applications; see [P2,4] and references therein.

It is well-known that Schauder first considered certain compactness of a continuous function instead of compactness of its domain. Therefore, it seems to be natural to ask whether the Fan-Browder theorem holds for compact maps.

Actually, Ben-El-Mechaiekh [B1,2] raised the following for the case $X = D$ is a convex subset of a topological vector space and $\Gamma_M = \operatorname{co} M$ for $M \in \langle X \rangle$.

Problem 1. *Does Theorem 2 hold if we assume T is compact instead of the compactness of X?*

This is still open. We discuss partial solutions of this problem.

The following is given in the proof of [PK2, Theorem 1] implicitly or [P8, Theorem 1(i)] explicitly.

Lemma 1. *Let Y be a Hausdorff space, $(X, D; \Gamma)$ a G-convex space, and $S : Y \multimap D$, $T : Y \multimap X$ maps satisfying*

(1) *for each $y \in Y$, $M \in \langle Sy \rangle$ implies $\Gamma_M \subset Ty$; and*

(2) $Y = \bigcup \{\operatorname{Int} S^- x : x \in D\}$.

Then $T \in \mathbb{C}^\kappa(Y, X) \subset \mathfrak{A}_c^\kappa(Y, X)$. More precisely, for any nonempty compact subset K of Y, $T|_K$ has a continuous selection $f : K \to X$; that is, $fy \in Ty$ for all $y \in K$, such that $f(K) \subset \Gamma_A$ for some $A \in \langle D \rangle$.

For a topological space Y and a G-convex space $(X, D; \Gamma)$, a map $T : Y \multimap X$ is called a Φ-*map* if there is a map $S : Y \multimap D$ satisfying (1) and (2) of Lemma 1.

From Theorem 1 and Lemma 1, we have the following:

Theorem 3. *Let (X, Γ) be a Hausdorff G-convex space, and $T : X \multimap X$ a Φ-map. If T is compact, then T^n has a fixed point for $n \geq 2$.*

Proof. By Lemma 1, we have $T \in \mathbb{C}^\kappa(X, X)$ and hence $T^m \in \mathbb{C}_c^\kappa(X, X) \subset \mathfrak{A}_c^\kappa(X, X)$ for $m \geq 1$. Note that T^m is compact. Let $S : X \multimap X$ be the companion map of T satisfying (2.1) and (2.2). Now we apply Theorem 1 with $X = D = Y$, $F = T^m$, $K = \overline{T^m(X)}$, and replacing S and T by $\operatorname{Int} S^-$ and T^-, respectively. Then

(1) for each $x \in X$, $\operatorname{Int} S^- x$ is open in X;

(2) for each $y \in X$, $M \in \langle (\operatorname{Int} S^-)^- y \rangle \subset \langle Sy \rangle$ implies $\Gamma_M \subset Ty$ by (2.1);

(3) for each $y \in \overline{T^m(X)} = K$, there exists an $x \in X$ such that $y \in \operatorname{Int} S^- x$ by (2.2), whence we have $K \subset (\operatorname{Int} S^-)(X)$; and

(4) $T^m(X) \backslash K = \emptyset$.

Therefore, all of the requirements of Theorem 1 for case (ii) are satisfied. Hence, we have a coincidence point $x_0 \in X$ of T^m and T^-; that is, there exists a $y_0 \in T^m x_0 \cap T^- x_0$. Since $x_0 \in T y_0$ and $y_0 \in T^m x_0$, we have $x_0 \in T^{m+1} x_0$. This completes our proof.

A particular form of Theorem 3 was obtained by Ben-El-Mechaiekh [B2] for a convex subset X of a Hausdorff topological vector space (simply, t.v.s.).

Now Problem 1 can be restated as follows:

Problem 2. *Does T have a fixed point under the hypothesis of Theorem 3 (for the case $T = S$ with open fibers)?*

We discuss partial solutions of this problem.

First of all, if X is a convex subset of a locally convex t.v.s., then Problem 2 is affirmative; see Ben-El-Mechaiekh *et al.* [BDG, Theorem 3.2; B2, Theorem 3] and Park [P3, Corollary 2.2]. In these works, it is used that $T \in \mathbb{C}_c^\pi$; see Corollary 9 below.

Recently, we obtained a more general result as follows:

Theorem 4. [P6,7, Theorem 1] *Let E be a t.v.s., X a convex subset of E, and $T \in \mathfrak{A}_c^\kappa(X, X)$ a compact map. If $\overline{T(X)}$ is admissible* (*in the sense of Klee*), *then T has a fixed point.*

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - hx \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s., l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, and many other t.v.s. are admissible; see [P6,7] and references therein.

From Theorem 4, we deduce the following:

Theorem 5. *Let X be a convex subset of a t.v.s. E and $T : X \multimap X$ a compact Φ-map; that is, there exists a map $S : X \multimap X$ such that*

(5.1) *for each $x \in X$, $\operatorname{co} Sx \subset Tx$; and*

(5.2) $X = \bigcup\{\operatorname{Int} S^{-}y : y \in X\}$.

If $\overline{T(X)}$ is admissible, then T has a fixed point.

Proof. Since X is a G-convex space (X, Γ) with $\Gamma_M = \operatorname{co} M$ for $M \in \langle X \rangle$, by Lemma 1, we have $T \in \mathfrak{A}_c^\kappa(X, X)$. Since $\overline{T(X)}$ is admissible and T is compact, by Theorem 4, it has a fixed point.

Recall that, more early, we obtained the following partial solution of Problem 2 :

Theorem 6. [P5, Theorem 2′] *Under the hypothesis of Theorem 5 with $S = T$, if $\overline{T(X)}$ is convexly totally bounded* (*in the sense of Idzik* [I]) *in E, then T has a fixed point.*

There have appeared many open questions on the concepts of admissible sets and convexly totally bounded sets related to the well-known Schauder conjecture; see [I], [W].

In view of Theorems 3-6, we can raise a general form of the Schauder conjecture:

Problem 3. *Does a nonempty convex subset of a (metrizable) t.v.s. have the fixed point property for comapct continuous functions? or for* $\mathfrak{A}_c^\kappa$*?*

A G-convex space $(X, D; \Gamma)$ is called a Φ-*space* if X is a Hausdorff uniform space and, for each entourage V, there exists a Φ-map $T : X \multimap X$ such that $\mathrm{Gr}(T) \subset V$.

The following is recently known by the author:

Lemma 2. [P8, Theorem 2] *If* $(X, D; \Gamma)$ *is a* Φ*-space, then any continuous compact function* $g : X \to X$ *has a fixed point.*

Since every nonempty convex subset of a locally convex t.v.s. is a Φ-space, Lemma 2 includes the fixed point theorems due to Brouwer, Schauder, and Tychonoff.

From now on, we consider a particular subclass of G-convex spaces originated from Horvath [H1,2]:

A G-convex space $(X, D; \Gamma)$ is called a C-*space* if each Γ_A is contractible (or more generally, n-connected for all $n \geq 0$) and, for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. A C-space (X, Γ) is an *LC-space* [H1] (or a *locally C-convex space* [T]) if X is a Hausdorff uniform space and if there exists a basis $\{V_i\}_{i \in I}$ for symmetric entourages such that for each $i \in I$, $\{x \in X : E \cap V_i[x] \neq \emptyset\}$ is Γ-convex whenever $E \subset X$ is Γ-convex, where

$$V_i[x] = \{x' \in X : (x, x') \in V_i\}.$$

A C-space (X, Γ) is an *LC-metric space* if X is equipped with a metric d such that for any $\varepsilon > 0$, the set $\{x \in X : d(x, A) < \varepsilon\}$ is Γ-convex whenever A is Γ-convex in X and open balls in (X, d) are Γ-convex.

We have the following selection theorem:

Lemma 3. [P8, Theorem 8] *Let* Y *be a paracompact space,* $(X, D; \Gamma)$ *a* C*-space, and* $T : Y \multimap X$ *a* Φ*-map. Then* T *has a continuous selection.*

Note that Lemma 3 slightly extends Horvath [H1, Theorem 3.2].

From Lemmas 2 and 3, we have the following:

Theorem 7. *Let* $(X, D; \Gamma)$ *be a paracompact* C*-space. If it is also a* Φ*-space, then any compact* Φ*-map* $T : X \multimap X$ *has a fixed point.*

Proof. By Lemma 3, T has a continuous selection $f : X \to X$. Since $f(X) \subset \overline{T(X)}$, f is a compact continuous function. Therefore, by Lemma 2, f has a fixed point $x_0 = fx_0 \in Tx_0$. This completes our proof.

In view of Lemma 3, if the first part of Problem 3 were affirmatively solved, then the problem of Ben-El-Mechaiekh would be affirmative; that is, Theorems 5 or 6 holds without the restriction that $\overline{T(X)}$ is admissible or convexly totally bounded, respectively.

More precisely, we have the following:

Theorem 8. *Let E be a t.v.s. whose nonempty convex subsets have the fixed point property for compact continuous single-valued selfmaps. Let X be a nonempty convex subset of E and $T : X \multimap X$ a Φ-map. If T is compact, then T has a fixed point.*

Proof. Let $K = \operatorname{co} \overline{T(X)}$. Then $K \subset X$ since $\overline{T(X)} \subset X$ and X is convex. Also K is σ-compact (see Fournier and Granas [FG]) and hence K is Lindelöf. Since K is regular as a subset of a t.v.s., we know that K is paracompact. Therefore, by Lemma 3, T has a continuous selection $f : K \to K$. Since T is compact, so is f. By hypothesis, f has a fixed point $x_0 \in K$; that is, $x_0 = fx_0 \in Tx_0$. This completes our proof.

In view of Theorem 8, Ben-El-Mechaiekh's problem is heavily depends on the Schauder conjecture.

From any of Theorems 4-8, we have the following well-known result [BDG, B2, P3]:

Corollary 9. *Let X be a nonempty convex subset of a locally convex t.v.s. Then any compact Φ-map $T : X \multimap X$ has a fixed point.*

Recently, Tarafdar [T] obtained the following:

Lemma 4. [T, Theorem 2.1] *If (X, Γ) is an LC-space and $F : X \multimap X$ is a compact continuous multimap with nonempty closed Γ-convex values, then F has a fixed point.*

From Lemmas 3 and 4, we have the following:

Theorem 10. *Let (X, Γ) be a paracompact LC-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$. Then any compact Φ-map $T : X \multimap X$ has a fixed point.*

Proof. By Lemma 3, T has a continuous selection $f : X \to X$. Since T is compact, f is a compact continuous function. Moreover, for each $x \in X$, fx is closed since X is Hausdorff, and fx is Γ-convex since $\Gamma_{\{fx\}} = \{fx\}$. Therefore, by Lemma 4, f has a fixed point. This completes our proof.

The following selection theorem is a generalization of a well-known result of Michael:

Lemma 5. [BO, Theorem 3] *Let Y be a paracompact space, (X, Γ) a complete LC-metric space, $Z \subset Y$ with $\dim_Y Z \leq 0$, and $T : Y \multimap X$ a lower semicontinuous map with nonempty closed values such that Ty is Γ-convex for $y \notin Z$. Then T has a continuous selection $f : Y \to X$.*

Combining Lemmas 4 and 5, we have the following:

Theorem 11. *Let (X, Γ) be a complete LC-metric space such that $\Gamma_{\{x\}} = \{x\}$ for $x \in X$, $Z \subset X$ with $\dim_X Z \leq 0$, and $T : X \multimap X$ a compact lower semiconitnuous map with nonempty closed values such that Tx is Γ-convex for $x \notin Z$. Then T has a fixed point.*

Proof. By Lemma 5 with $X = Y$, T has a continuous selection $f : X \to X$. Since T is compact, so is f. Now by Lemma 4, f has a fixed point. This completes our proof.

Since a map with open fibers is lower semicontinuous, from Theorem 11, we have one more particular solution of Problem 2 as follows:

Theorem 12. *Let (X, Γ) and Z be the same as in Theorem 11. Let $T : X \multimap X$ be a compact map such that*

(11.1) *for each $x \in X$, Tx is nonempty and closed;*
(11.2) *for each $x \notin Z$, Tx is a Γ-convex; and*
(11.3) *for each $y \in X$, $T^{-}y$ is open.*

Then T has a fixed point.

Note that any closed convex subset of a Banach space or a completely metrizable locally convex t.v.s. and any hyperconvex space is a complete LC-metric space; see [H2].

Conclusion. Theorems 4–8, 11, and 12 are partial solutions to Problems 1 or 2.

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