J. Korean Math. Soc. **36** (1999), No. 4, pp. 813–828

COINCIDENCE THEOREMS ON A PRODUCT OF GENERALIZED CONVEX SPACES AND APPLICATIONS TO EQUILIBRIA

SEHIE PARK AND HOONJOO KIM

ABSTRACT. In this paper, we give a Peleg type KKM theorem on G-convex spaces and using this, we obtain a coincidence theorem. First, these results are applied to a whole intersection property, a section property, and an analytic alternative for multimaps. Secondly, these are used to prove existence theorems of equilibrium points in qualitative games with preference correspondences and in n-person games with constraint and preference correspondences for non-paracompact setting of commodity spaces.

0. Introduction

In the last two decades there have appeared many generalizations of the classical Arrow-Debreu result on existence of the Walrasian equilibria in various directions. For compact commodity spaces, Gale and Mas-Colell [7] proved existence of a competitive equilibrium, and Borglin and Keiding [3] proved a new existence theorem for an abstract economy, which was generalized to an abstract economy with infinite number of agents by Yennelis and Prabhakar [25]. And Tarafdar [24] showed existence of abstract economies whose commodity spaces are H-spaces.

Furthermore Tan-Yuan [22], Ding-Tarafdar [6], and Ding-Tan [5] proved equilibrium theorems for non-compact generalized games with correspondences defined on a non-compact (but paracompact) commodity space. And Tan-Yu-Yuan [23] obtained existence theorems of

---

Received April 17, 1999.

1991 Mathematics Subject Classification: Primary 54H25, 90D06, 46N10.

Key words and phrases: G-convex space, Γ-convex set, G-KKM, abstract economy, correspondence, n-person game, equilibrium, qualitative game, Γ-quasiconcave.

This paper was supported in part by the Non-directed Research Fund, Korea Research Foundation, 1997, and by KOSEF-981-0102-013-2.

equilibrium points of n-person games with constraint and preference correspondences on non-compact H-spaces.

In 1967, motivated by the search for equilibrium points in non-cooperative games, Peleg [21] established an extension of the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem [11]. Since then Peleg's lemma has been widely used in the framework of game theory in order to prove existence results concerning different solution concepts, like the bargaining set and the kernel.

Recently, the authors introduced generalized convex (or G-convex) spaces which are adequate to establish theories on fixed points, coincidence points, KKM maps, equilibrium problems, best approximations, and many others. For details, see [17-20].

In this paper, we give a Peleg type KKM theorem on G-convex spaces, which was used to derive a coincidence theorem. First, these results are applied to a whole intersection property, a geometric lemma, and an analytic alternative for multimaps. Secondly, these are used to prove existence theorems of equilibrium points in qualitative games with preference correspondences and in n-person games with constraint and preference correspondences for non-paracompact setting of commodity spaces.

1. Preliminaries

A *multimap* (*map* or *correspondence*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with *values* $Fx \subset Y$ for $x \in X$ and *fibers* $F^-y = \{x \in X : y \in Fx\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. As usual, the set $\{(x,y) \in X \times Y : y \in Fx\}$ is called the *graph* of F and denoted by F. A map $F : X \multimap Y$ is *compact* provided $F(X)$ is contained in a compact subset of Y. For any $B \subset Y$, the (*lower*) *inverse* and (*upper*) *inverse* of B under F are defined by

$$F^-(B) = \{x \in X : Fx \cap B \neq \emptyset\} \text{ and } F^+(B) = \{x \in X : Fx \subset B\},$$

resp. The (*lower*) *inverse* of $F : X \multimap Y$ is the map $F^- : Y \multimap X$ defined by $x \in F^-y$ if and only if $y \in Fx$. Given two maps $F : X \multimap Y$ and $G : Y \multimap Z$, their *composite* $GF : X \multimap Z$ is defined by $(GF)x = G(Fx)$ for each $x \in X$.

For topological spaces X and Y, a map $F : X \multimap Y$ is *lower semi-continuous* (l.s.c.) if, for each open set $B \subset Y$, $F^{-}(B)$ is open in X.

For a nonempty set D, let $\langle D \rangle$ denote the set of all nonempty finite subsets of D. For a set A, let $|A|$ denote the cardinality of A. Let Δ_n denote the standard n-simplex; that is,

$$\Delta_n = \left\{ u \in \mathbf{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u) e_i,\ \lambda_i(u) \geq 0,\ \sum_{i=1}^{n+1} \lambda_i(u) = 1 \right\},$$

where e_i is the i-th unit vector in $\mathbf{R}^{n+1}$. For each $u = \sum_{i=1}^{n+1} \lambda_i(u) e_i$ in Δ_n, the $(n+1)$-tuple $(\lambda_1(u), \cdots, \lambda_{n+1}(u))$ is called the *barycentric coordinate* of $u \in \Delta_n$.

Let X be a set (in a vector space) and D a nonempty subset of X. Then (X, D) is called a *convex space* if convex hulls of any nonempty finite subset of D is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. A subset A of X is said to be D-*convex* if, for each $N \in \langle D \rangle$, $N \subset A$ implies $\operatorname{co} N \subset A$, where co denotes the convex hull. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [12].

Let X be a topological space. A *c-structure* on X is given by a map $F : \langle X \rangle \multimap X$ such that

(1) for all $A \in \langle X \rangle$, $F(A)$ is nonempty and contractible; and
(2) for all $A, B \in \langle X \rangle$, $A \subset B$ implies $F(A) \subset F(B)$.

A pair (X, F) is then called a *c-space* by Horvath [8,9] and an H-*space* by Bardaro and Ceppitelli [1].

A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ with $|A| = n+1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_1, a_2, \cdots, a_{n+1}\}$, then $\Delta_J = \{u \in \Delta_n : \sum_j \lambda_j(u) = 1, a_j \in J\}$. For details on G-convex spaces, see [17-20].

We may write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D \rangle$. Note that Γ_A does not need to contain A for $A \in \langle D \rangle$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by (X, Γ). For simplicity, we assume that $D \subset X$ in this paper. For an $(X, D; \Gamma)$, a subset C of X is said to be Γ-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. For a nonempty subset S of X, the Γ-*convex hull* of S, $\Gamma\text{-co}\, S$, is defined by

$$\Gamma\text{-co}\, S = \bigcap \{Y : S \subset Y \subset X \text{ and } Y \text{ is } \Gamma\text{-convex}\}.$$

Any convex space (X, D) becomes a G-convex space $(X, D; \Gamma)$ by putting $\Gamma_A = \text{co}\, A$. An H-space (X, F) is a G-convex space (X, Γ). In fact, by putting $\Gamma_A = F(A)$ for each $A \in \langle X \rangle$ with $|A| = n + 1$, there exists a continuous map $\phi_A : \Delta_n \to X$ such that for all $J \subset A$, $\phi_A(\Delta_J) \subset F(J)$ by Horvath [8, Theorem 1].

The other major examples of G-convex spaces are convex subsets of a t.v.s., metric spaces with Michael's convex structure, S-contractible spaces, Horvath's pseudo-convex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, and Joó's pseudoconvex spaces ; for the literature, see [17-20]. Recently, we gave new examples of G-convex spaces and, simultaneously, showed that some abstract convexities of other authors are simple particular examples of our G-convexity; see [16]. Such examples are L-spaces of Ben-El-Mechaiekh *et al.*, continuous images of G-convex spaces, generalized H-spaces of Verma or Stachó, and mc-spaces of Llinares. Moreover, Ben-El-Mechaiekh *et al.* [2] gave examples of G-convex spaces (X, Γ) as follows: B'-simplicial convexity, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces. More recently, it is noted that every almost convex subset of a topological vector space has a G-convexity.

For a convex space (X, D), a map $F : D \multimap X$ is called a *KKM map* if $\text{co}\, N \subset F(N)$ for each $N \in \langle D \rangle$.

For a G-convex space $(X, D; \Gamma)$, a map $F : D \multimap X$ is called a *G-KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$.

2. Coincidence Theorems

In this section, for each $i \in I = \{1, \cdots, n\}$, let $(X_i, D_i; \Gamma_i)$ be a G-convex space, Y a topological space, $X = \Pi_{i \in I} X_i$, and $D = \Pi_{i \in I} D_i$. We may write $\Gamma_i(A_i) = \Gamma_{A_i}$ for each $A_i \in \langle D_i \rangle$ and $i \in I$.

First we need the following generalization of the classical KKM theorem due to Peleg [21]:

LEMMA. *For each $i \in I$, let C_i^j, $j = 1, \cdots, m_i + 1$, be closed subsets of $\Pi_{i\in I}\Delta_{m_i}$ such that for each $A_i \subset \{1, \cdots, m_i + 1\}$ and $i \in I$,*

$$\Delta_{m_1} \times \cdots \times \Delta_{A_i} \times \cdots \times \Delta_{m_n} \subset \bigcup_{j\in A_i} C_i^j,$$

where Δ_{A_i} denotes the face of Δ_{m_i} corresponding to A_i. Then

$$\bigcap_{i\in I} \bigcap_{j=1}^{m_i+1} C_i^j \neq \emptyset.$$

Now we generalize Lemma to G-convex spaces:

THEOREM 1. *For each $i \in I$, let $(X_i, D_i; \Gamma_i)$ be a compact G-convex space, $G_i : D_i \multimap Y$ and $F : X \multimap Y$ multimaps such that*

(1.1) *for each $i \in I$, and $x_i \in D_i$, $F^+G_ix_i$ is closed; and*
(1.2) *for each $A = \Pi_{i\in I}A_i \in \langle D\rangle$,*

$$\Pi_{i\in I}\Gamma_{A_i} \subset \bigcap_{i\in I}\bigcup\{F^+G_ix_i : x_i \in A_i\}.$$

Then there exists an $x \in X$ such that $Fx \subset \bigcap_{i\in I}\bigcap_{x_i\in D_i} G_ix_i$.

Proof. For any $B = \Pi_{i\in I}B_i \in \langle D\rangle$, there exists a continuous function $f_i : \Delta_{B_i} \to \Gamma_{B_i}$ such that $f_i(\Delta_{A_i}) \subset \Gamma_{A_i}$ for each $A_i \subset B_i$ and $i \in I$, where Δ_{B_i} is a $(|B_i| - 1)$-simplex and Δ_{A_i} denotes the face of $\Delta_{|B_i|-1}$ corresponding to A_i. Let $fz = \Pi_{i\in I}f_iz$ for each $z \in \Pi_{i\in I}\Delta_{B_i}$. For any $i \in I$ and $A_i \subset B_i$, the set $f^-(F^+G_ix_i)$ is closed for all $x_i \in A_i$ since f is continuous. Moreover,

$$\begin{aligned}\prod_{i\in I}\Delta_{A_i} &\subset f^-f(\prod_{i\in I}\Delta_{A_i}) = f^-\big(\prod_{i\in I}f_i(\Delta_{A_i})\big)\\ &\subset f^-(\prod_{i\in I}\Gamma_{A_i}) \subset f^-(\bigcap_{i\in I}\bigcup\{F^+G_ix_i : x_i \in A_i\})\\ &= \bigcap_{i\in I}\bigcup\{f^-F^+G_ix_i : x_i \in A_i\}.\end{aligned}$$

Hence by Lemma,

$$f^-(\bigcap_{i\in I}\bigcap_{x_i\in B_i} F^+G_ix_i) = \bigcap_{i\in I}\bigcap_{x_i\in B_i} f^-F^+G_ix_i \neq \emptyset;$$

that is, $f^-(\bigcap_{i\in I}\bigcap_{x_i\in D_i} F^+G_ix_i) \neq \emptyset$ since X is compact. Therefore, we have $\bigcap_{i\in I}\bigcap_{x_i\in D_i} F^+G_ix_i \neq \emptyset$ and hence, there exists an $x \in X$ such that $Fx \subset \bigcap_{i\in I}\bigcap_{x_i\in D_i} G_ix_i$. □

REMARK. For an H-space X and $F = 1_X$, Theorem 1 reduces to Marchi and Martínez-Legaz [14, Corollary 5].

The following is a generalization of Theorem 1 on non-compact setting:

THEOREM 2. *Let K be a nonempty compact subset of Y. For each $i \in I$, suppose $G_i : D_i \multimap Y$ and $F : X \multimap Y$ satisfy (1.2) and the following:*

- (2.1) *for each $i \in I$ and $x_i \in D_i$, $F^+G_ix_i$ and G_ix_i are compactly closed; and*
- (2.2) *for each $A = \Pi_{i\in I}A_i \in \langle D\rangle$, there exists a compact Γ-convex subset $L = \Pi_{i\in I}L_i$ of X such that L_i contains A_i for each $i \in I$ and*

$$L \cap \bigcap_{i\in I}\bigcap\{F^+G_ix_i : x_i \in L_i \cap D_i\} \subset F^+K.$$

Then we have

$$\overline{F(X)} \cap K \cap \bigcap_{i\in I}\bigcap_{x_i\in D_i} G_ix_i \neq \emptyset.$$

Proof. Suppose the conclusion does not hold. Since $\overline{F(X)} \cap K$ is compact, there exists an $A = \Pi_{i\in I}A_i \in \langle D\rangle$ such that

$$\overline{F(X)} \cap K \subset \bigcup_{i\in I}\bigcup_{x_i\in A_i} (Y\backslash G_ix_i).$$

For the set L in (2.2), we have

$$L \cap \bigcap_{i\in I}\bigcap\{F^+G_ix_i : x_i \in L_i \cap D_i\} \cap F^+K = \emptyset.$$

But

$$L \cap \bigcap_{i\in I}\bigcap\{F^+G_ix_i : x_i \in L_i \cap D_i\} \subset F^+K,$$

by (2.2), we have

$$L \cap \bigcap_{i\in I} \bigcap \{F^+ G_i x_i : x_i \in L_i \cap D_i\} = \emptyset. \tag{2.3}$$

Define $H_i x_i = F^+ G_i x_i \cap L$ for $x_i \in L_i \cap D_i$ and $i \in I$. Note that $L_i \cap D_i \neq \emptyset$ for $i \in I$. Consider $(L_i, L_i \cap D_i, L, H_i, 1_L)$ instead of (X_i, D_i, Y, G_i, F) in Theorem 1. Then all of the requirements of Theorem 1 are satisfied. Hence

$$L \cap \bigcap_{i\in I} \bigcap_{x_i \in L_i \cap D_i} F^+ G_i x_i = \bigcap_{i\in I} \bigcap_{x_i \in L_i \cap D_i} H_i x_i \neq \emptyset,$$

which contradicts (2.3). □

REMARKS. 1. Condition (2.1) is satisfied if we assume one of the following:

(i) F is l.s.c. and $G_i x_i$ is closed for each $x_i \in D_i$ and $i \in I$.
(ii) F is a compact-valued continuous multimap and $G_i x_i$ is compactly closed for each $x_i \in D_i$ and $i \in I$.
(iii) $F = t : X \to Y$ is a single-valued continuous function and $G_i x_i$ is compactly closed for each $x_i \in D_i$ and $i \in I$.

For details, see [15].

2. For an H-space X, $m = 1$, and a single-valued continuous function F, Theorem 2 reduces to Chang and Yang [4, Lemma 1.3].

From Theorem 2, we obtain the following:

THEOREM 3. *Let K be a nonempty compact subset of Y. For $i \in I$, let $S_i : D_i \multimap Y$, $T_i : X_i \multimap Y$ be multimaps, and $t : X \to Y$ a continuous function satisfying*

(3.1) *for each $i \in I$ and $x_i \in D_i$, $S_i x_i$ is compactly open in Y;*
(3.2) *for each $i \in I$ and $y \in t(X)$, $A_i \in \langle S_i^- y\rangle$ implies $\Gamma_{A_i} \subset T_i^- y$;*
(3.3) *for all $y \in \overline{t(X)} \cap K$, $y \in S_i(D_i)$ for some $i \in I$; and*
(3.4) *for each $A = \Pi_{i\in I} A_i \in \langle D\rangle$, there exists a compact Γ-convex subset $L = \Pi_{i\in I} L_i$ of X such that L_i contains A_i for each $i \in I$ and*

$$t(L)\backslash K \subset \bigcup_{i\in I} S_i(L_i \cap D_i).$$

Then there exists an $\tilde{x} = \Pi_{i\in I}\tilde{x}_i \in X$ *such that* $t\tilde{x} \in T_i\tilde{x}_i$ *for some* $i \in I$.

Proof. For each $i \in I$, let $G_i : D_i \multimap Y$ be a multimap defined by $G_i x_i = Y \backslash S_i x_i$ for each $x_i \in D_i$. Clearly G_i has compactly closed values. By (3.3),

$$\overline{t(X)} \cap K \cap \bigcap_{i\in I} \bigcap_{x_i\in D_i} G_i x_i = \overline{t(X)} \cap K \cap \{Y \backslash \bigcup_{i\in I} S_i(D_i)\} = \emptyset.$$

There exists an $A = \Pi_{i\in I}A_i \in \langle D\rangle$ such that $t(\Pi_{i\in I}\Gamma_{A_i}) \not\subset \bigcap_{i\in I} G_i(A_i)$ by Theorem 2. Take $\tilde{x} \in \Pi_{i\in I}\Gamma_{A_i}$ such that $t\tilde{x} = \tilde{y} \notin \bigcap_{i\in I} G_i(A_i)$. For some $i \in I$, $\tilde{y} \notin G_i(A_i)$ or $\tilde{y} \in \bigcap_{x_i\in A_i} S_i x_i$; that is, $A_i \subset S_i^- \tilde{y}$. Hence, there exists an $\tilde{x}_i \in \Gamma_{A_i} \subset T_i^- \tilde{y}$ by (3.2). Therefore $\tilde{y} \in T_i\tilde{x}_i$ for some $i \in I$. □

REMARK. For a compact H-space $X = Y = D$ and $t = 1_X$, Theorem 3 reduces to Marchi and Martínez-Legaz [14, Theorem 6].

From Theorem 2, we have another whole intersection property as follows:

THEOREM 4. *Let* K *be a nonempty compact subset of* Y. *For each* $i \in I$, *let* $G_i : D_i \multimap Y$, $H_i : X_i \multimap Y$ *be multimaps, and* $t : X \to Y$ *a continuous function such that*

(4.1) *for each* $i \in I$ *and* $x_i \in D_i$, $G_i x_i$ *is compactly closed;*
(4.2) *for each* $x = \Pi_{i\in I}x_i \in X$, $tx \in \bigcap_{i\in I} H_i x_i$;
(4.3) *for each* $y \in t(X)$ *and* $i \in I$, $A_i \in \langle D_i \backslash G_i^- y\rangle$ *implies* $\Gamma_{A_i} \subset X_i \backslash H_i^- y$; *and*
(4.4) *for each* $A = \Pi_{i\in I}A_i \in \langle D\rangle$, *there exists a compact* Γ-*convex subset* $L = \Pi_{i\in I}L_i$ *of* X *such that* L_i *contains* A_i *for each* $i \in I$ *and*

$$t(L) \cap \bigcap_{i\in I} \bigcap \{G_i x_i : x_i \in L_i \cap D_i\} \subset K.$$

Then

$$\overline{t(X)} \cap K \cap \bigcap_{i\in I} \bigcap_{x_i\in D_i} G_i x_i \neq \emptyset.$$

Proof. Suppose that there exists an $A = \Pi_{i\in I}A_i \in \langle D \rangle$ such that $t(\Pi_{i\in I}\Gamma_{A_i}) \not\subset \bigcap_{i\in I} G_i(A_i)$; that is, there exists an $\tilde{x} = \Pi_{i\in I}\tilde{x}_i \in \Pi_{i\in I}\Gamma_{A_i}$ such that $\tilde{y} = t\tilde{x} \notin \bigcap_{i\in I} G_i(A_i)$. In other words, $A_i \in \langle D_i \backslash G_i^- \tilde{y} \rangle$ for some $i \in I$. By (4.3), $\Gamma_{A_i} \subset X_i \backslash H_i^- \tilde{y}$, and since $\tilde{x}_i \in \Gamma_{A_i}$, we have $\tilde{x}_i \notin H_i^- \tilde{y}$ or $\tilde{y} \notin H_i \tilde{x}_i$, which contradicts (4.2). So all of the requirements of Theorem 2 are satisfied for $F = t$, and hence the conclusion holds. □

3. Section Properties

For each $i \in I = \{1, \cdots, n\}$, let $(X_i, D_i; \Gamma_i)$, Y, X, and D be the same as in Section 2.

We now deduce a section property or a geometric form of Theorem 4.

THEOREM 5. *Let K be a nonempty compact subset of Y, $t : X \to Y$ a continuous function, $N_i \subset X_i \times Y$ and $M_i \subset D_i \times Y$ for each $i \in I$. Suppose that*

(5.1) *for each $i \in I$ and $x_i \in D_i$, $\{y \in Y : (x_i, y) \in M_i\}$ is compactly closed in Y;*
(5.2) *for each $x = \Pi_{i\in I}x_i \in X$, $(x_i, tx) \in N_i$ for all $i \in I$;*
(5.3) *for each $y \in t(X)$ and $i \in I$, $A_i \in \langle \{x_i \in D_i : (x_i, y) \notin M_i\} \rangle$ implies $\Gamma_{A_i} \subset \{x_i \in X_i : (x_i, y) \notin N_i\}$; and*
(5.4) *for each $A = \Pi_{i\in I}A_i \in \langle D \rangle$, there exists a compact Γ-convex subset $L = \Pi_{i\in I}L_i$ of X such that L_i contains A_i for each $i \in I$ and*

$$t(L) \cap \bigcap_{i\in I} \bigcap_{x_i \in L_i \cap D_i} \{y \in Y : (x_i, y) \in M_i\} \subset K.$$

Then there exists a $\tilde{y} \in \overline{t(X)}$ such that $\Pi_{i\in I}\big(D_i \times \{\tilde{y}\}\big) \subset \Pi_{i\in I}M_i$.

Proof. For each $i \in I$ and $x_i \in D_i$, let

$$G_i x_i = \{y \in Y : (x_i, y) \in M_i\},$$

which is compactly closed by (5.1). Moreover, for each $i \in I$ and $x_i \in X_i$, let $H_i x_i = \{y \in Y : (x_i, y) \in N_i\}$. Then (5.2)–(5.4) imply

(4.2)–(4.4), resp. Therefore, we have

$$\overline{t(X)} \cap K \cap \bigcap_{i\in I} \bigcap_{x_i\in D_i} G_i x_i \neq \emptyset.$$

Hence there exists a $\tilde{y} \in \overline{t(X)} \cap K$ such that $\tilde{y} \in \bigcap_{i\in I} \bigcap_{x_i\in D_i} G_i x_i$; that is, $\Pi_{i\in I}(D_i \times \{\tilde{y}\}) \subset \Pi_{i\in I} M_i$. □

REMARK. Theorem 5 generalizes [14, Corollary 8].

The following is a reformulation of Theorem 3.

THEOREM 6. *Let K be a nonempty compact subset of Y. For each $i \in I$, let $N_i, M_i \subset Z_i$ be sets, $t : X \to Y$ a continuous function, $g_i : D_i \times Y \to Z_i$ and $h_i : X_i \times Y \to Z_i$ functions. Suppose that*

- (6.1) *for each $i \in I$ and $x_i \in D_i$, $\{y \in Y : g_i(x_i, y) \in N_i\}$ is compactly open in Y;*
- (6.2) *for each $y \in t(X)$ and $i \in I$, $A_i \in \langle\{x_i \in D_i : g_i(x_i, y) \in N_i\}\rangle$ implies $\Gamma_{A_i} \subset \{x_i \in X_i : h_i(x_i, y) \in M_i\}$; and*
- (6.3) *for each $A = \Pi_{i\in I} A_i \in \langle D\rangle$, there exists a compact Γ-convex subset $L = \Pi_{i\in I} L_i$ of X such that L_i contains A_i for each $i \in I$ and*

$$t(L)\backslash K \subset \bigcup_{i\in I} \bigcup_{x_i\in L_i\cap D_i} \{y \in Y : g_i(x_i, y) \in N_i\}.$$

Then either

- (a) *there exists a $\tilde{y} \in \overline{t(X)} \cap K$ such that $g_i(x_i, \tilde{y}) \notin N_i$ for all $i \in I$ and $x_i \in D_i$; or*
- (b) *there exists an $\tilde{x} = \Pi_{i\in I}\tilde{x}_i \in X$ such that $h_i(\tilde{x}_i, t\tilde{x}) \in M_i$ for some $i \in I$.*

Proof. For each $i \in I$, consider the multimaps $S_i : D_i \multimap Y$ and $T_i : X_i \multimap Y$ given by

$$S_i x_i = \{y \in Y : g_i(x_i, y) \in N_i\} \text{ for } x_i \in D_i,$$

and

$$T_i x_i = \{y \in Y : h_i(x_i, y) \in M_i\} \text{ for } x_i \in X_i.$$

Then (6.1)–(6.3) imply (3.1), (3.2), and (3.4), resp. Suppose that (a) does not hold. Then for each $y \in \overline{t(X)} \cap K$, there exists an $i \in I$ and an $x_i \in D_i$ such that $g_i(x_i, y) \in N_i$; that is, $\overline{t(X)} \cap K \subset \bigcup_{i \in I} S_i(D_i)$. Hence (3.3) holds. Therefore, by Theorem 3, there exists an $\tilde{x} = \Pi_{i \in I} \tilde{x}_i \in X$ and an $i \in I$ such that $t\tilde{x} \in T_i \tilde{x}_i$; that is, (b) holds. □

From Theorem 6, we have the following analytic alternative, which is a basis of various minimax inequalities:

THEOREM 7. *Let K be a nonempty compact subset of Y. For each $i \in I$, let $\alpha_i, \beta_i \in \mathbf{R}$, $t : X \to Y$ a continuous function, $g_i : D_i \times Y \to \overline{\mathbf{R}}$, and $h_i : X_i \times Y \to \overline{\mathbf{R}}$ extended real-valued functions. Suppose that*

(7.1) *for each $i \in I$ and $x_i \in D_i$, $\{y \in Y : g_i(x_i, y) > \alpha_i\}$ is compactly open;*

(7.2) *for each $y \in t(X)$ and $i \in I$, $A_i \in \langle \{x_i \in D_i : g_i(x_i, y) > \alpha_i\} \rangle$ implies $\Gamma_{A_i} \subset \{x_i \in X_i : h_i(x_i, y) > \beta_i\}$; and*

(7.3) *for each $A = \Pi_{i \in I} A_i \in \langle D \rangle$, there exists a compact Γ-convex subset $L = \Pi_{i \in I} L_i$ of X such that L_i contains A_i for each $i \in I$ and*

$$t(L) \backslash K \subset \bigcup_{i \in I} \bigcup_{x_i \in L_i \cap D_i} \{y \in Y : g_i(x_i, y) > \alpha_i\}.$$

Then either

(a) *there exists a $\tilde{y} \in \overline{t(X)} \cap K$ such that $g_i(x_i, \tilde{y}) \le \alpha_i$ for all $i \in I$ and $x_i \in D_i$; or*

(b) *there exists an $\tilde{x} = \Pi_{i \in I} \tilde{x}_i \in X$ such that $h_i(\tilde{x}_i, t\tilde{x}) > \beta_i$ for some $i \in I$.*

Proof. Put $Z_i = \overline{\mathbf{R}}$, $N_i = (\alpha_i, \infty]$ and $M_i = (\beta_i, \infty]$ for each $i \in I$ in Theorem 6. □

4. Equilibrium Existence Theorems

If X is a topological space, $(Y, D; \Gamma)$ is a G-convex space, and $F : X \multimap Y$ is a multimap, then $\Gamma\text{-co}\, F, \overline{F} : X \multimap Y$ are multimaps defined by $(\Gamma\text{-co}\, F)x = \Gamma\text{-co}(Fx)$, and $\overline{F}x = \{y \in Y : (x, y) \in \text{cl}_{X \times Y} F\}$ for each $x \in X$, resp.

An *abstract economy* (or *generalized game*) is a family of quadruples $(X_i, A_i, B_i, P_i)_{i\in I}$ where I is a (finite or infinite) set of agents (players) such that for each $i \in I$, X_i is a nonempty subset of a topological space, $X = \Pi_{i\in I} X_i$ and $A_i, B_i : X \multimap X_i$ are *constraint correspondences*, and $P_i : X \multimap X_i$ is a *preference correspondence*. When $I = \{1, \dots, n\}$, $(X_i, A_i, B_i, P_i)_{i\in I}$ is also called an *n-person game*. An *equilibrium* of $(X_i, A_i, B_i, P_i)_{i\in I}$ is a point $x = (x_i)_{i\in I} \in X$ such that for each $i \in I, x_i \in \overline{B}_i x$ and $A_i x \cap P_i x = \emptyset$.

From now on we only consider $I = \{1, \dots, n\}$. Denote $X^i = \Pi_{j\in I\setminus\{i\}} X_i$, $x = (x_1, \dots, x_n) \in X$, $x^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X^i$, $(x^i, y_i) = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \in X$.

Following the notion of Gale and Mas-Colell [7], the collection $(X_i, P_i)_{i\in I}$ will be called a *qualitative game*. A point $x \in X$ is said to be an *equilibrium* of the game $(X_i, P_i)_{i\in I}$ if $P_i x = \emptyset$ for all $i \in I$. See also [13, 23].

THEOREM 8. *For each $i \in I$, let $(X_i, P_i)_{i\in I}$ be a qualitative game such that $(X_i; \Gamma_i)$ is a G-convex space, and $P_i : X \multimap X_i$ a correspondence. Suppose that*

(8.1) *for each $i \in I$ and $x \in X$, $x_i \notin \Gamma\text{-co}P_i x$;*
(8.2) *for each $i \in I$ and $y_i \in X_i$, $P_i^- y_i$ is compactly open; and*
(8.3) *for each $A = \Pi_{i\in I} A_i \in \langle X \rangle$, there exists a compact Γ-convex subset $L = \Pi_{i\in I} L_i$ of X such that L_i contains A_i for each $i \in I$ and*

$$L\backslash K \subset \bigcup_{i\in I} P_i^-(L_i).$$

Then $(X_i, P_i)_{i\in I}$ has an equilibrium point $\tilde{x} \in K$; that is, $P_i\tilde{x} = \emptyset$ for all $i \in I$.

Proof. Suppose that $(X_i, P_i)_{i\in I}$ has no equilibrium point in K; that is, for each $x \in K$, $P_i x \neq \emptyset$ for some $i \in I$. For $i \in I$, put $S_i, T_i : X_i \multimap X$ by $S_i x_i = P_i^- x_i$ and $T_i x_i = (\Gamma\text{-co}P_i)^- x_i$, resp. By the assumption, $K \subset \bigcup_{i\in I} P_i^{-1}(X_i) = \bigcup_{i\in I} S_i(X_i)$; that is, (3.3) holds for $t = 1_X$ and $D_i = X_i$ for $i \in I$. Since other conditions of Theorem 3 hold also, there exists an $\tilde{x} \in X$ such that $\tilde{x} \in T_i\tilde{x}_i$ for some $i \in I$ which implies $\tilde{x}_i \in T_i^- \tilde{x} = \Gamma\text{-co}P_i\tilde{x}$. This contradicts (8.1). □

THEOREM 9. *Let* $(X_i, A_i, B_i, P_i)_{i\in I}$ *be an* n*-person game such that for each* $i \in I$, $(X_i;\Gamma_i)$ *is a* G*-convex space,* $A_i, B_i, P_i : X \multimap X_i$ *correspondences, and* K *a nonempty compact subset of* X. *Suppose that*

(9.1) *for each* $i \in I$ *and* $x \in X$, A_ix *is nonempty and* Γ-co$A_ix \subset \overline{B}_ix$;
(9.2) *for each* $x \in X$, $\overline{B}_ix = \mathrm{cl}_{X_i}B_ix$ *for each* $i \in I$;
(9.3) *for each* $i \in I$ *and* $x_i \in X_i$, $A_i^-x_i$ *and* $P_i^-x_i$ *is compactly open;*
(9.4) $x_i \notin \Gamma$-coP_ix *for all* $x \in X$ *and* $i \in I$; *and*
(9.5) *for each* $N = \Pi_{i\in I}N_i \in \langle X\rangle$, *there exists a compact* Γ*-convex subset* $L = \Pi_{i\in I}L_i$ *of* X *such that* L_i *contains* N_i *for each* $i \in I$ *and*
$$L\backslash K \subset \bigcup_{i\in I}(A_i \cap P_i)^-(L_i).$$

Then $(X_i, A_i, B_i, P_i)_{i\in I}$ *has an equilibrium point* $\tilde{x} \in X$; *that is, for each* $i \in I$, $\tilde{x}_i \in \mathrm{cl}_{X_i}B_i\tilde{x}$ *and* $A_i\tilde{x} \cap P_i\tilde{x} = \emptyset$.

Proof. For each $i \in I$, let $F_i = \{x \in X : x_i \notin \mathrm{cl}_{X_i}B_ix\}$. Then F_i is open in X by (9.2). Define $\varphi_i : X \multimap X_i$ by

$$\varphi_ix = \begin{cases} A_ix \cap P_ix & \text{if } x \notin F_i \\ A_ix & \text{if } x \in F_i. \end{cases}$$

Fix any $i \in I$. If $x \in F_i$, Γ-co$\,\varphi_ix = G$-co$A_ix \subset \overline{B}_ix$, and since $x_i \notin \mathrm{cl}_{X_i}B_ix = \overline{B}_ix$, $x_i \notin \Gamma$-co$\,\varphi_ix$. If $x \notin F_i$, $x_i \notin \Gamma$-coP_ix implies that $x_i \notin \Gamma$-co$\,(A_ix \cap P_ix) = \Gamma$-co$\,\varphi_ix$. Further, $\varphi_i^-y = A_i^-y \cap (F_i \cup P_i^-y)$ is compactly open by (9.3). By (9.5) and the definition of φ_i, φ_i satisfies (8.3). By Theorem 8, there exists an $\tilde{x} \in K$ such that $\varphi_i\tilde{x} = \emptyset$ for all $i \in I$. Since $A_i\tilde{x} \neq \emptyset$ for all $i \in I$, $\tilde{x}_i \in \mathrm{cl}_{X_i}B_i\tilde{x}$ and $A_i\tilde{x} \cap P_i\tilde{x} = \emptyset$ for all $i \in I$. □

Let $(X;\Gamma)$ be a G-convex space. A function $f : X \to \mathbf{R}$ is said to be Γ-*quasiconcave* if for each $A \in \langle X\rangle$ and $x \in \Gamma_A$, $fx \geq \min_{y\in A} fy$.

THEOREM 10. *For each* $i \in I$, *let* $(X_i;\Gamma_i)$ *be a* G*-convex space,* $F_i = X^i \multimap X_i$ *a correspondence, and* $u_i : X \multimap \mathbf{R}$ *a function. Suppose that*

(10.1) *for each* $i \in I$ *and* $x^i \in X^i$, F_ix^i *is nonempty and* Γ*-convex in* X_i;

(10.2) *for each* $i \in I$ *and* $x^i \in X^i$, $\overline{F}_i x^i = \mathrm{cl}_{X_i} F_i x^i$;
(10.3) *for each* $y_i \in X_i$, $F_i^- y_i$ *is compactly open in* X^i;
(10.4) u_i *is continuous in* x *and* Γ*-quasiconcave in* x_i; *and*
(10.5) *there exist a nonempty compact subset* K *of* X *and an* $M \in \langle X \rangle$ *such that for each* $x \in X \backslash K$, *there exists a* $y \in M$ *satisfying* $y_i \in F_i x^i$ *for all* $i \in I$ *and* $u_i(x^i, x_i) < u_i(x^i, y_i)$ *for some* $i \in I$.

Then the generalized game $(X_i, F_i, u_i)_{i \in I}$ *has an equilibrium point* $\tilde{x} \in X$; *that is,* $\tilde{x}_i \in \overline{F}_i \tilde{x}^i$ *and* $u_i(\tilde{x}^i, \tilde{x}_i) = \sup_{y_i \in F_i \tilde{x}^i} u_i(\tilde{x}^i, y_i)$.

Proof. For $i \in I$, define $A_i, P_i : X \multimap X_i$ by $A_i x = F_i x^i$ and $P_i x = \{y_i \in X_i : u_i(x^i, y_i) > u_i x\}$ for all $x = (x^i, x_i) \in X$. Clearly (9.1) holds with $A_i = B_i$ for $i \in I$. Fix an $x \in X$ and $i \in I$. It is clear that $\mathrm{cl}_{X_i} A_i x \subset \overline{A}_i x$. And if $y_i \in \overline{A}_i x$, then $(x^i, y_i) \in \overline{F}_i$ and

$$y_i \in \overline{F}_i x^i = \mathrm{cl}_{X_i} F_i x^i = \mathrm{cl}_{X_i} A_i x.$$

Hence (10.2) implies (9.2). For $i \in I$ and $x \in X$, $x_i \notin P_i x$. Now we show that $P_i x$ is Γ-convex for $i \in I$ and $x \in X$. Indeed, for each $N_i \in \langle P_i x \rangle$ and $z_i \in \Gamma_{N_i}$,

$$u_i(x^i, z_i) \geq \min_{y_i \in N_i} u_i(x^i, y_i) > u_i(x^i, x_i) = u_i x;$$

that is, $z_i \in P_i x$, hence $\Gamma_{N_i} \subset P_i x$. For each $y_i \in X_i$ and $i \in I$, $A_i^- y_i = \{x \in X : x^i \in F_i^- y_i\} = F_i^- y_i \times X_i$ and $P_i^- y_i = \{x \in X : u_i x < u_i(x^i, y_i)\}$ are compactly open in X by (10.3), and (10.4). And (10.5) implies (9.5). Hence, there exists an $\tilde{x} \in X$ such that $\tilde{x}_i \in \mathrm{cl}_{X_i} A_i \tilde{x} = \overline{F}_i \tilde{x}^i$ and $A_i \tilde{x} \cap P_i \tilde{x} = \emptyset$; that is, $\tilde{x}_i \in \overline{F}_i \tilde{x}^i$ and $u_i \tilde{x} = \sup_{y_i \in F_i \tilde{x}^i} u_i(\tilde{x}^i, y_i)$. □

REMARK. Theorem 10 is a correct and generalized form of Huang [10, Theorem 2]. In particular, (10.2) is essential as the following example shows:

EXAMPLE. Let $X_1 = [0,1]$, $X_2 = [1,2]$, $F_1 : [1,2] \multimap [0,1]$ and $F_2 : [0,1] \multimap [1,2]$ be defined by

$$F_1(x_2) = \begin{cases} [\frac{1}{2}, 1] & x_2 = 1 \\ \{\frac{1}{2}\} & x_2 = \frac{3}{2} \\ [0,1] & x_2 \neq 1, \frac{3}{2} \end{cases}$$

and

$$F_2(x_1) = \begin{cases} \{\frac{3}{2}\} & x_1 = 0 \\ [\frac{7}{4}, 2] & x_1 = \frac{1}{2} \\ [1,2] & x_1 \neq 0, \frac{1}{2} \end{cases}$$

for each $x_1 \in [0,1]$ and $x_2 \in [1,2]$, resp. And let $u_1, u_2 : [0,1] \times [1,2] \to \mathbb{R}$ be defined by

$$u_1(x_1, x_2) = -x_1^2$$

and

$$u_2(x_1, x_2) = -x_2^2$$

for each $(x_1, x_2) \in [0,1] \times [1,2]$, resp. Then F_1, F_2, u_1, u_2 satisfy (10.1), (10.3) and (10.4). But $\overline{F_1}(1) = [0,1] \neq \text{cl}_{X_1} F_1(1) = [\frac{1}{2}, 1]$. And $(X_i, F_i, u_i)_{i=1,2}$ has no equilibrium point.

References

[1] C. Bardaro and R. Ceppitelli, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities*, J. Math. Anal. Appl. **132** (1988), 484–490.

[2] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J. V. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–151.

[3] A. Borglin and H. Keiding, *Existence of equilibrium action and of equilibrium: A note on the 'new' existence theorems*, J. Math. Econom. **3** (1976), 313–316.

[4] S. S. Chang and L. Yang, *Section theorems on H-spaces with applications*, J. Math. Anal. Appl. **179** (1993) 214–231.

[5] X. P. Ding and K. K. Tan, *On equilibria of noncompact generalized games*, J. Math. Anal. Appl. **177** (1993), 226–238.

[6] X. P. Ding and E. Tarafdar, *Fixed point theorems and equilibria on noncompact generalized games*, Fixed Point Theory and Applications, (K.-K. Tan, ed.), World Scientific, River Edge, NJ, 1992, pp. 80–96.

[7] D. Gale and A. Mas-Colell, *On the role of complete, transitive preferences in equilibrium theory*, Equilibrium and Disequilibrium in Economic Theory (G. Schwodiauer, ed.), Reider, Rordrecht, 1978, pp. 7–14.

[8] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.

[9] ———, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2** (1993), 253–269.

[10] Y. Huang, *Fixed point theorems with an application in generalized games*, J. Math. Anal. Appl. **186** (1994), 634–642.

[11] B. Knaster, K. Kuratowski, S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.

[12] M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.

[13] M. Lassonde and C. Schenkel, *KKM principle, fixed points and Nash equilibria*, J. Math. Anal. Appl. **164** (1992), 542–548.

[14] E. Marchi and J. E. Martínez-Legaz, *A generalization of Fan-Browder's fixed point theorem and its applications*, Top. Meth. Nonlin. Anal. **2** (1993), 277–291.

[15] S. Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, ed.), World Scientific, River Edge, NJ, 1992, pp. 248–277.

[16] ———, *New subclasses of generalized convex spaces*, Proc. Internat. Conf. on Math. Anal. Appl. (Chinju, 1998) **1-A** (1999), 65–72.

[17] S. Park and H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci. Seoul Nat. U. **18** (1993), 1–21.

[18] ———, *Coincidence theorems of admissible maps on generalized convex spaces*, J. Math. Anal. Appl. **197** (1996), 173–187.

[19] ———, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl. **209** (1997), 551–571.

[20] ———, *Generalizations of the KKM type theorems on generalized convex spaces*, Ind. J. Pure Appl. Math. **29** (1998), 121–132.

[21] B. Peleg, *Equilibrium points for open acyclic relations*, Canad. J. Math. **19** (1967), 366–369.

[22] K. K. Tan and X. Z. Yuan, *A minimax inequality with applications to existence of equilibrium points*, Bull. Austral. Math. Soc. **47** (1993), 487–503.

[23] K. K. Tan, J. Yu and X. Z. Yuan, *Some new minimax inequalities and applications to existence of equilibria in H-spaces*, Nonlinear Analysis, TMA **24** (1995), 1457–1470.

[24] E. Tarafdar, *Fixed point theorems in H-spaces and equilibrium points of abstract economies*, J. Austral. Math. Soc. (A) **53** (1992), 252–260.

[25] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom. **12** (1983), 233–245.

Department of Mathematics
Seoul National University
Seoul 151–742, Korea
E-mail: shpark@math.snu.ac.kr

Department of Computer Science
Daebul University
Young-Am 526–890, Korea
E-mail: hoonjoo@math.snu.ac.kr