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Fixed Points of Lower Semicontinuous Multimaps in LC-Metric Spaces*

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Ricceri's results on fixed points of lower semicontinuous multimaps in Banach spaces are extended to sharpened forms for complete LC-metric spaces due to Horvath.

1. INTRODUCTION

The story began with the celebrated works of John von Neumann on game theory in 1928 [11] and mathematical economics in 1937 [12]. In 1941, Kakutani [7] obtained a fixed point theorem for upper semicontinuous multimaps and applied it to give simple proofs of the minimax theorem [11] and the intersection theorem [12] of von Neumann. Later it was known that the intersection theorem is equivalent to Kakutani's theorem. Since then a large number of works concerning fixed point theorems and minimax theorems in various directions have appeared. For the literature, see [13].

In 1985, Ricceri [21] pointed out that some drawbacks with regard to upper semicontinuous multimaps having closed convex values disappear in the case of lower semicontinuous multimaps. Actually, he obtained a coincidence theorem, from which he deduced a fixed point theorem for lower semicontinuous multimaps that are allowed to have many nonclosed or nonconvex values. Those theorems are applied to new alternative and minimax theorems in [21]. Note that Ricceri [21] was mainly concerned

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with the convexity of subsets of Banach spaces, and based his argument on a selection theorem due to Michael.

On the other hand, the convexity of subsets of topological vector spaces is extended to convex spaces by Lassonde [9], to C-spaces by Horvath [2−5], and to G-convex spaces by Park [17−20]. It is known that the KKM theory, fixed point theory, and other equilibrium results are now well developed in those abstract convexities without linear structures. Especially, Ben-El-Mechaiekh and Oudadess [1] obtained far-reaching generalizations of some of Michael's selection theorems to complete LC-metric spaces which are C-spaces of a particular type including closed convex subsets of Banach spaces or completely metrizable locally convex topological vector spaces, hyperconvex metric spaces, and others.

In this paper, based on a selection theorem in [1], we obtain generalizations of most results of Ricceri [21] to sharpened forms for complete LC-metric spaces.

2. PRELIMINARIES

A *multimap* or *map* $F: X \multimap Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y, that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) = \{x \in X: y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{F(x): x \in A\}$, and for any $B \subset Y$, let $F^-(B) = \{x \in X: F(x) \cap B \neq \emptyset\}$.

For topological spaces X and Y, a map $F: X \multimap Y$ is said to be *compact* if $F(X)$ is contained in a compact subset of Y; *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c. If $X \subset Y$, we put $\mathrm{Fix}(F) = \{x \in X: x \in F(x)\}$.

If S is a topological space and $A \subset S$, $\dim_S A \leq 0$ means that the covering dimension of T is ≤ 0 for every set $T \subset A$ which is closed in S; see Hurewicz and Wallman [6].

Let Y be a topological space and $\langle Y \rangle$ the set of all nonempty finite subsets of Y.

A *C-space* (Y, Γ) consists of a topological space Y and a multimap $\Gamma: \langle Y \rangle \multimap Y$ such that

(1) for each $A \in \langle Y \rangle$, $\Gamma_A := \Gamma(A)$ is ω-connected (that is, n-connected for all $n \geq 0$); and

(2) for each $A, B \in \langle Y \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A subset Z of Y is said to be Γ-*convex* (or a *C-set*) if $\Gamma_A \subset Z$ for each $A \in \langle Z \rangle$.

An *LC-metric space* (Y, d, Γ) is a C-space if Y is equipped with a metric d such that for any $\varepsilon > 0$, the set $\{y \in Y: d(y, Z) < \varepsilon\}$ is Γ-convex whenever $Z \subset Y$ is Γ-convex and all open balls are Γ-convex.

For details on C-spaces, see Horvath [2–5].

In this paper, we are mainly concerned with complete LC-metric spaces: obvious examples of them are (closed convex subsets of) Banach spaces, completely metrizable locally convex topological vector spaces, hyperconvex metric spaces, and others; see Horvath [5]. We assume that all topologies in this paper are Hausdorff.

The following is due to Ben-El-Mechaiekh and Oudadess [1, Corollary 6(A)(iii)]:

LEMMA 1. *Let X be a paracompact space, Y a complete LC-metric space where singletons are Γ-convex, $Z \subset X$ with $\dim_X Z \leq 0$, D a countable subset of X, and $\Phi: X \multimap Y$ a l.s.c. map such that $\Phi(x)$ is closed for $x \notin D$, and $\overline{\Phi(x)}$ is Γ-convex for $x \notin Z$. Then Φ has a continuous selection.*

For related results, see I. Kim [8].

The following is due to Tarafdar [23, Theorem 2.1 and Corollary 2.2]:

LEMMA 2. *Let (X, d, Γ) be an LC-metric space such that singletons are Γ-convex. Then each compact continuous function $f: X \to X$ has a fixed point.*

The following is due to Michael [10, Proposition 2.3]:

LEMMA 3. *If $\Phi: X \multimap Y$ is l.s.c. and if $\Psi: X \multimap Y$ is such that $\overline{\Psi(x)} = \overline{\Phi(x)}$ for every $x \in X$, then Ψ is l.s.c.*

We also need the following due to Ricceri [21, Proposition 1]:

LEMMA 4. *Let S, Σ be two topological spaces, and $\Phi: S \multimap \Sigma$ a multimap such that the set $\{\sigma \in \Sigma: \Phi^{-}(\sigma)$ is open$\}$ is dense in Σ and $\overline{\Phi(s)} = \overline{\operatorname{Int}(\Phi(s))}$ for every $s \in S$. Then Φ is l.s.c.*

A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \varnothing$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

It is known that the space $\mathbb{C}(E)$ of all continuous real functions on a Stonian space E (that is, extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathbb{C}(E)$ for some Stonian space E. Therefore, $(\mathbb{R}^n, \|\cdot\|_\infty)$, l^∞, and L^∞ are concrete examples of hyperconvex spaces.

Horvath [5, Theorem 9] obtained the following:

LEMMA 5. *Any hyperconvex space H is a complete LC-metric space with*

$$\Gamma_A = \bigcap \{B : B \text{ is a closed ball containing } A\}$$

for each $A \in \langle H \rangle$.

3. COINCIDENCE AND FIXED POINT THEOREMS

In this section, we obtain our main results.

In the sequel, the closure operation $\overline{}$ for a subset of an LC-metric space (X, d, Γ) is always with respect to the metric d.

THEOREM 1. *Let (X, d, Γ), (Y, d_1, Γ_1) be two complete LC-metric spaces such that singletons are Γ-convex and Γ_1-convex, resp. Let τ be a topology on X weaker than the d-topology such that (X, τ) is paracompact and τ_1 a topology on Y weaker than the d_1-topology. Let C, Z be two subsets of X and D, W two subsets of Y with C, D countable and* $\dim_{(X,\tau)} Z \le 0$, $\dim_{(Y,\tau_1)} W \le 0$. *Suppose that $F : (X, \tau) \multimap (Y, d_1)$ and $G : (Y, \tau_1) \multimap (X, d)$ satisfy the following:*

(1) *F is (τ, d_1)-l.s.c. and compact, $F(x)$ is d_1-closed for $x \in X \setminus C$, and $\overline{F(x)}$ is Γ_1-convex for $x \in X \setminus Z$; and*

(2) *G is (τ_1, d)-l.s.c., $G(y)$ is d-closed for $y \in Y \setminus D$, and $\overline{G(y)}$ is Γ-convex for $y \in Y \setminus W$.*

Then there exists $(x^*, y^*) \in X \times Y$ such that $x^* \in G(y^*)$ and $y^* \in F(x^*)$.

Proof. Since (X, τ) is paracompact, by Lemma 1, there exists a (τ, d_1)-continuous selection $f : X \to Y$ of $F : X \multimap Y$. Note that $(\overline{f(X)}, d_1)$ is compact and τ_1 is weaker than d_1-topology, and hence $(\overline{f(X)}, \tau_1)$ is also compact. Therefore, by Lemma 1 again, there exists a (τ_1, d)-continuous selection $g : \overline{f(X)} \to X$ of $G|_{\overline{f(X)}} : \overline{f(X)} \multimap X$. Let us consider the composition

$$(X, d) \xrightarrow{1_X} (X, \tau) \xrightarrow{f} \left(\overline{f(X)}, d_1\right) \to \left(\overline{f(X)}, \tau_1\right) \xrightarrow{g} (X, d).$$

From this, the composition $g \circ f : (X, \tau) \to (X, d)$ can be regarded as the continuous function $g \circ f : (X, d) \to (X, d)$, which is compact since so is $\overline{f(X)}$. Therefore, by Lemma 2, $g \circ f$ has a fixed point $x^* \in X$; that is, $x^* \in (g \circ f)(x^*)$. Hence, there exists a $y^* \in Y$ such that $x^* = g(y^*) \in G(y^*)$ and $y^* = f(x^*) \in F(x^*)$. This completes our proof.

Note that Theorem 1 generalizes Ricceri [21, Theorem 1], where (U, d, Γ), (V, d_1, Γ_1) are Banach spaces with d, d_1 metrics induced by norms, $X \subset U$, $Y \subset V$, and Γ, Γ_1 usual convex hulls under the extra restriction that (X, τ) and (Y, τ_1) are compact and $\overline{\mathrm{co}\, F(X)} \subset Y$, $\overline{\mathrm{co}\, G(Y)} \subset X$. In fact, replacing X and Y in Theorem 1 by $\overline{\mathrm{co}\, G(Y)}$ and $\overline{\mathrm{co}\, F(X)}$, resp., Theorem 1 reduces to [21, Theorem 1].

From Theorem 1, we have the following fixed point theorem:

THEOREM 2. *Let $(X, d, \Gamma), \tau, C, Z$ be the same as in Theorem 1 and $F: X \multimap X$ a (τ, d)-l.s.c. compact multimap such that $F(x)$ is d_1-closed for $x \in X \setminus C$, $\overline{F(x)}$ is Γ-convex for $x \in X \setminus Z$. Then $\mathrm{Fix}(F) \neq \emptyset$. If, in addition, for every $x \in \mathrm{Fix}(F)$, x is a d-accumulation point of $F(x)$, then $\mathrm{Fix}(F)$ is uncountable.*

Proof. As in the proof of Theorem 1, we have $\mathrm{Fix}(F) \neq \emptyset$. Now suppose that for every $x \in \mathrm{Fix}(F)$, x is a d-accumulation point of $F(x)$. Consider the multimap $\tilde{F}: X \multimap X$ defined by

$$\tilde{F}(x) = \begin{cases} F(x) \setminus \{x\} & \text{if } x \in \mathrm{Fix}(F), \\ F(x) & \text{if } x \in X \setminus \mathrm{Fix}(F). \end{cases}$$

Since $\overline{\tilde{F}(x)} = \overline{F(x)}$ for every $x \in X$, $\tilde{F}$ is (τ, d)-l.s.c. by Lemma 3. On the other hand, $\tilde{F}(x)$ is d-closed for every $x \in X \setminus (C \cup \mathrm{Fix}(F))$ and $\overline{\tilde{F}(X)}$ is Γ-convex for every $x \in X \setminus Z$. Therefore, if $\mathrm{Fix}(F)$ is countable, it follows from the first half of Theorem 2 that $\mathrm{Fix}(\tilde{F}) \neq \emptyset$. This contradicts the definition of $\tilde{F}$. This completes our proof.

Note that Theorem 2 generalizes Ricceri [21, Theorem 2], where (U, d, Γ) is a Banach space with d of the metric induced by the norm, $X \subset U$, $\overline{\mathrm{co}\, F(X)} \subset X$, and Γ the convex hull.

Since a multimap with open fibers is l.s.c., from Theorem 2, we have the following Fan−Browder type fixed point theorem:

COROLLARY 1. *Let $(X, d, \Gamma), \tau, C, Z$ be the same as in Theorem 1 and $F: (X, \tau) \multimap (X, d)$ a compact multimap such that*

(1) *$F(x)$ is d-closed for each $x \in X \setminus C$;*

(2) *$\overline{F(x)}$ is Γ-convex for each $x \in X \setminus Z$; and*

(3) *$F^{-}(y)$ is τ-open.*

Then the conclusion of Theorem 2 holds.

Note that some related results to Corollary 1 were given in [14, 15]. Actually, Corollary 1 generalizes [14, Theorem 12], but not the corresponding one in [15].

From Theorem 2 and Lemma 5, we have the following:

COROLLARY 2. *Let* (H, d) *be a hyperconvex space,* C *a countable subset of* H*, and* Z *a subset of* H *such that* $\dim_H Z \leq 0$. *Let* $F\colon H \multimap H$ *be a compact l.s.c. map such that* $F(x)$ *is closed for* $x \in H \setminus C$ *and* $\overline{F(x)}$ *is* Γ*-convex for* $x \in H \setminus Z$. *Then the conclusion of Theorem* 2 *holds.*

Note that Corollary 2 extends a part of [16, Theorem 7].

4. MINIMAX AND ALTERNATIVE THEOREMS

A real function φ defined on a C-space is said to be *quasi-convex* (resp., *quasi-concave*) if, for every $t \in \mathbf{R}$, the set $\varphi^{-1}(]-\infty, t[)$ (resp., $\varphi^{-1}(]t, +\infty[)$) is Γ-convex.

From Theorem 1, we obtain the following minimax theorem:

THEOREM 3. *Let* (X, d, Γ), (Y, d_1, Γ_1), τ, τ_1, Z, W *be the same as in Theorem* 1. *Suppose that* (X, τ) *and* (Y, τ_1) *are compact. Let* $f\colon X \times Y \to \mathbf{R}$ *be a function such that*

(1) *the function* $f(x, \cdot)$ *is* τ_1*-l.s.c. for every* $x \in X$ *and quasi-convex for every* $x \in X \setminus Z$;

(2) *the function* $f(\cdot, y)$ *is* τ*-u.s.c. for every* $y \in Y$ *and quasi-concave for every* $y \in Y \setminus W$.

Then we have

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. Let α and β be the left and right hand side, resp. It is clear that $\alpha \leq \beta$. Suppose that $\alpha < \beta$. Let us fix a, b such that $\alpha < a < b < \beta$. For every $\bar{x} \in X$ and $\bar{y} \in Y$, let

$$F(\bar{x}) = \left\{\overline{y \in Y\colon f(\bar{x}, y) < a}\right\}, \qquad \Phi(\bar{x}) = \{y \in Y\colon f(\bar{x}, y) \leq a\},$$

$$G(\bar{y}) = \left\{\overline{x \in X\colon f(x, \bar{y}) > b}\right\}, \qquad \Psi(\bar{y}) = \{x \in X\colon f(x, \bar{y}) \geq b\}.$$

Clearly $\Phi(x) \supset F(x) \neq \emptyset$, $\Psi(y) \supset G(y) \neq \emptyset$ for $x \in X$, $y \in Y$. It can be checked that F and G satisfy the hypotheses of Theorem 1. In fact, consider the map $V\colon X \multimap Y$ defined by $V(\bar{x}) = \{y \in Y\colon f(\bar{x}, y) < a\}$ for $\bar{x} \in X$. Since $f(\cdot, y)$ is τ-u.s.c. for every $y \in Y$ by (2), V has τ-open fibers and hence V is (τ, d_1)-l.s.c. Since $F(\bar{x}) = \overline{V(\bar{x})}$ for all $\bar{x} \in X$, F is (τ, d_1)-l.s.c. by Lemma 3. Similarly, G is (τ_1, d)-l.s.c. by (1). The other conditions are easily checked. Therefore, there exists $(x^*, y^*) \in X \times Y$

such that $x^* \in \Psi(y^*)$ and $y^* \in \Phi(x^*)$; that is, $b \le f(x^*, y^*) \le a$, a contradiction.

Note that Theorem 3 generalizes Ricceri [21, Theorem 3], where (U, d, Γ), (V, d_1, Γ_1) are Banach spaces with d, d_1 metrics induced by norms, $X \subset U$, $Y \subset V$, and Γ, Γ_1 the usual convex hulls. Moreover, Theorem 3 is a far-reaching generalization of the von Neumann minimax theorem [11].

From Theorem 2, we deduce the following analytic alternative:

THEOREM 4. *Let* $(X, d, \Gamma), \tau, Z$ *be the same as in Theorem* 3, *and* $f\colon X \times X \to \mathbf{R}$ *a function such that*

(1) *the function* $f(x, \cdot)$ *is* d-*l.s.c. for every* $x \in X$ *and quasi-convex for every* $x \in X \setminus Z$; *and*

(2) *the function* $f(\cdot, y)$ *is* τ-*u.s.c. for every* $y \in X$.

Then, for any τ-*l.s.c. real function* $\varphi\colon X \to \mathbf{R}$, *at least one of the following holds*:

(i) *There exists an* $x_0 \in X$ *such that* $\inf_{y \in X} f(x_0, y) \ge \varphi(x_0)$.

(ii) *There exists an* $x^* \in X$ *such that* $f(x^*, x^*) \le \varphi(x^*)$.

Proof. Define $F, \Phi\colon X \multimap X$ by

$$F(x) = \overline{\{y \in X : f(x, y) < \varphi(x)\}} \text{ and } \Phi(x) = \{y \in X : f(x, y) \le \varphi(x)\}$$

for $x \in X$. Suppose that (i) does not hold. Then $\emptyset \ne F(x) \subset \Phi(x)$ for every $x \in X$ by (1). As in the proof of Theorem 3, F is (τ, d)-l.s.c. since $f(\cdot, y) - \varphi$ is τ-u.s.c. Other requirements of Theorem 2 are easily checked. Therefore, by Theorem 2, there exists an $x^* \in X$ such that $x^* \in F(x^*) \subset \Phi(x^*)$, whence (ii) holds.

Note that Theorem 4 reduces to Ricceri [21, Theorem 4] for a closed convex subset X of a Banach space $(U, \|\cdot\|)$.

From Theorem 2, we have one more analytic alternative:

THEOREM 5. *Let* $(X, d, \Gamma), \tau, Z$ *be the same as in Theorem* 3, *with* X *not a singleton, and let* $f\colon X \times X \to \mathbf{R}$ *be a function such that*

(1) *the function* $f(x, \cdot)$ *is* d-*continuous for every* $x \in X$ *and quasi-convex for every* $x \in X \setminus Z$;

(2) *the set* $\{y \in X : f(\cdot, y)$ *is* τ-*u.s.c.*$\}$ *is* d-*dense in* X.

Then, for any τ-*l.s.c. real function* $\varphi\colon X \to \mathbf{R}$, *at least one of the following holds*:

(i) *There exists an* $x_0 \in X$ *such that* $\inf_{y \in X} f(x_0, y) \ge \varphi(x_0)$.

(ii) *The set* $\{x \in X : f(x, x) \le \varphi(x)\}$ *is uncountable.*

Proof. Suppose that statement (i) does not hold. For every $x \in X$, put

$$\Phi(x) = \{y \in X : f(x,y) \leq \varphi(x)\}, \quad H(x) = \{y \in X : f(x,y) < \varphi(x)\}$$

and

$$F(x) = \overline{H(x)}.$$

Note that, by (1), $H(x)$ is d-open in X and $F(x) \subset \Phi(x)$. By (2), the set $\{y \in X : H^{-}(y)$ is τ-open$\}$ is d-dense in X. Therefore, by Lemma 4, the multimaps H and F are (τ, d)-l.s.c. Since X is not a singleton, every $x \in \text{Fix}(F)$ is a d-accumulation point of $F(x)$. Hence, by Theorem 2, $\text{Fix}(\Phi)$ is uncountable. Therefore, statement (ii) holds.

Note that Theorem 5 reduces to Ricceri [21, Theorem 5] for a closed convex subset X of a Banach space $(U, \|\cdot\|)$.

Until now, we generalized all of the results of Ricceri [21], except [21, Theorem 6], for which we have to study more in order to obtain its generalization or variation for C-spaces. Note that O. Naselli Ricceri [22] obtained a variation of [21, Theorem 2] which can be also extended like our Theorem 2.

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