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Fixed point theorems in hyperconvex metric spaces[1]

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1. Introduction

The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi [1] in 1956. In 1979, independently Sine [13] and Soardi [16] proved the fixed point property for nonexpansive maps on bounded hyperconvex spaces. Since then many interesting works have appeared for hyperconvex spaces. For example, see [2–8,14,15].

It is known that the space $\mathbb{C}(E)$ of all continuous real functions on a Stonian space E (extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathbb{C}(E)$ for some Stonian space E. Then $(\mathbf{R}^n, \|\cdot\|_\infty)$, l^∞ and L^∞ are concrete examples of hyperconvex spaces.

Until recently, the study of hyperconvex spaces was concentrated to the relationship with nonexpansive maps. However, recently, Khamsi [5] established the Knaster–Kuratowski–Mazurkiewicz theorem (in short, KKM theorem) for hyperconvex spaces and applied it to prove an analogue of Ky Fan's best approximation theorem extending the Brouwer and the Schauder fixed point theorems. This seems to be the first attempt to prove such results in a hyperconvex space setting.

In this paper, we obtain a Ky Fan type matching theorem for open covers, a coincidence theorem, a Fan–Browder type fixed point theorem, a Brouwer–Schauder–Rothe type fixed point theorem, and other results for hyperconvex spaces. Those are usually

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obtained for a convex space setting (see [9, 10]) or a generalized convex space setting (see [11, 12]).

Our arguments in this paper are based on the KKM theorem due to Khamsi [5, Theorem 4].

2. Preliminaries

We follow mainly Khamsi [5].

A metric space (H,d) is said to be *hyperconvex* if for any collection of points $\{x_\alpha\}$ of H and for any collection $\{r_\alpha\}$ of nonnegative reals such that $d(x_\alpha,x_\beta)\leq r_\alpha+r_\beta$, we have

$$\bigcap_\alpha B(x_\alpha,r_\alpha)\neq\emptyset.$$

Here $B(x,r)$ denotes the closed ball with center $x\in H$ and radius $r\geq 0$.

For any nonempty bounded subset A of H, its *convex hull* $\operatorname{co}A$ is defined by

$$\operatorname{co}A=\bigcap\{B\colon B \text{ is a closed ball containing } A\}.$$

Let $\mathscr{A}(H)=\{A\subset H\colon A=\operatorname{co}A\}$; i.e. $A\in\mathscr{A}(H)$ iff A is an intersection of balls. In this case we will say A is an *admissible* subset of H.

For any $X\subset H$, a multimap (or a map) $G:X\multimap H$ is called a *KKM map* if

$$\operatorname{co}\{x_1,\ldots,x_n\}\subset\bigcup_{i=1}^n G(x_i)$$

for any $x_1,\ldots,x_n\in X$.

The following KKM theorem is due to Khamsi [5, Theorem 4]:

KKM Theorem. *Let H be a hyperconvex space and $X\subset H$ a subset. Let $G:X\multimap H$ be a KKM-map such that $G(x)$ is closed for any $x\in X$ and $G(x_0)$ is compact for some $x_0\in X$. Then we have*

$$\bigcap_{x\in X} G(x)\neq\emptyset.$$

We will mainly use the particular case of KKM Theorem when H itself is compact.

For a multimap $G:X\multimap Y$, we denote $x\in G^{-1}(y)$ iff $y\in G(x)$ where $x\in X$ and $y\in Y$. Let $\mathbb{C}(X,Y)$ denote the class of single-valued continuous maps $f:X\to Y$.

3. Matching and coincidence theorems

From the KKM Theorem, we deduce the following Ky Fan type matching theorem for open covers:

Theorem 1. *Let H be a compact hyperconvex space, $X \subset H$, Y a topological space, and $A : X \multimap Y$ a map satisfying*
(i) *Ax is open for each $x \in X$, and*
(ii) *$A(X) = Y$.*
Then, for each $t \in \mathbb{C}(H, Y)$, there exists a nonempty finite subset $\{x_1, \dots, x_n\} \subset X$ and an $x_0 \in \mathrm{co}\{x_1, \dots, x_n\}$ such that $t(x_0) \in \bigcap_{i=1}^n A(x_i)$.

Proof. For each $x \in X$, let $F(x) = Y \backslash A(x)$ and let $G(x) = t^{-1}F(x)$. Suppose the conclusion is false. Then for every finite subset $\{x_1, \dots, x_n\} \subset X$, we have

$$t(\mathrm{co}\{x_1, \dots, x_n\}) \subset Y \backslash \bigcap_{i=1}^{n} A(x_i) = \bigcup_{i=1}^{n} F(x_i),$$

that is,

$$\mathrm{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} t^{-1} F(x_i) = \bigcup_{i=1}^{n} G(x_i).$$

Therefore, G is a closed-valued KKM map. Since H is compact, each Gx is compact. Then by KKM Theorem, we have $\bigcap \{G(x) \colon x \in X\} = \bigcap \{t^{-1}F(x) \colon x \in X\} \neq \emptyset$ and hence $\bigcap \{F(x) \colon x \in X\} \neq \emptyset$. But this means $\bigcup \{A(x) \colon x \in X\} = A(X) \neq Y$, a contradiction to our hypothesis. This completes our proof. □

From Theorem 1, we have the following Fan–Browder type coincidence theorem for hyperconvex spaces:

Theorem 2. *Let H, X, and Y be the same as in Theorem 1. Let $S : X \multimap Y$, $T : H \multimap Y$ be multifunctions satisfying the following*:
(a) *for each $x \in X$, $S(x) \subset T(x)$ and $S(x)$ is open*; *and*
(b) *for each $y \in Y$, $T^{-1}(y)$ is admissible*; *and*
(c) *$S(X) = Y$.*
Then for any $t \in \mathbb{C}(H, Y)$, there exists an $x_0 \in H$ such that $t(x_0) \in T(x_0)$.

Proof. Since (a) and (c) imply (i) and (ii) with $A = S$, by Theorem 1, there exist $\{x_1, \dots, x_n\} \subset X$ and $x_0 \in \mathrm{co}\{x_1, \dots, x_n\}$ such that $t(x_0) \in \bigcap_{i=1}^n S(x_i) \subset \bigcap_{i=1}^n T(x_i)$. We have $x_i \in T^{-1}t(x_0)$ for all i, and hence by (b), $\mathrm{co}\{x_1, \dots, x_n\} \subset T^{-1}t(x_0)$. In particular, $x_0 \in T^{-1}t(x_0)$, i.e. $t(x_0) \in T(x_0)$. This completes our proof. □

Remark. Theorems 1 and 2 are analogous to Park [9, Theorems 1 and 6] for convex spaces. (Note that in [9, Theorem 6], A should be $A : X \to 2^Y$ and $x_0 \in X$.)

4. Fixed point theorems

As an application of Theorem 2, we have the following form of the Fan–Browder type fixed point theorem for hyperconvex spaces:

Theorem 3. *Let H be a compact hyperconvex space and $T : H \multimap H$ a map such that*
(1) *for each $x \in H$, $T(x)$ is admissible*; *and*
(2) $H = \bigcup \{\mathrm{Int}\, T^{-1} y\colon\ y \in H\}$.
Then T has a fixed point $x_0 \in H$; i.e. $x_0 \in T(x_0)$.

Proof. Replace (H, X, Y, T) in Theorem 2 by (H, H, H, T^{-1}) and define $S : H \multimap H$ by $S(x) = \mathrm{Int}\, T^{-1}(x)$ for $x \in H$. Then, for each $x \in X$, $S(x) \subset T^{-1}(x)$ and $S(x)$ is open. Moreover, for each $y \in H$, $T(y)$ is admissible, and $H = S(H)$. Therefore, for $t = 1_H$, by Theorem 2, there exists an $x_0 \in H$ such that $x_0 \in T^{-1}(x_0)$ or $x_0 \in Tx_0$. This completes our proof. □

For convex spaces, Theorem 3 is well-known to be very useful in various problems.
As another application of Theorem 1, we obtain the following Fan type best approximation theorem:

Theorem 4. *Let H be a hyperconvex space, $X \in \mathscr{A}(H)$ compact, and $f \in \mathbb{C}(X, H)$. Then either f has a fixed point or there exists an $x_0 \in \mathrm{Bd}\, X$ such that*

$$0 < d(x_0, f(x_0)) \le d(x, f(x_0)) \quad for\ all\ x \in X.$$

Proof. Suppose that for each $y \in X$, there exists an $x \in X$ such that

$$d(x, f(y)) < d(y, f(y)).$$

Define $A : X \multimap X$ by

$$A(x) = \{y \in X\colon\ d(x, f(y)) < d(y, f(y))\}$$

for $x \in X$. Since f is continuous, $A(x)$ is open in X. Moreover, for each $y \in X$, we have $A^{-1}(y) \neq \emptyset$ by the assumption. Therefore $A(X) = X$. Now we apply Theorem 1 with $H = X = Y$ and $t = 1_X$. Then there exist $\{x_1, \ldots, x_n\} \subset X$ and $x_0 \in \mathrm{co}\{x_1, \ldots, x_n\}$ such that $x_0 \in \bigcap_{i=1}^n A(x_0)$; i.e. $d(x_i, f(x_0)) < d(x_0, f(x_0))$ for each i.
Let $\varepsilon > 0$ such that $d(x_i, f(x_0)) \le d(x_0, f(x_0)) - \varepsilon$. Then

$$x_i \in B(f(x_0), d(x_0, f(x_0)) - \varepsilon).$$

Since the closed ball is admissible and contains all x_i, we have

$$x_0 \in \mathrm{co}\{x_1, \ldots, x_n\} \subset B(f(x_0), d(x_0, f(x_0)) - \varepsilon).$$

This is a contradiction. Therefore, there exists an $x_0 \in X$ such that $d(x_0, f(x_0)) \le d(x, f(x_0))$ for all $x \in X$. If $d(x_0, f(x_0)) = 0$, then x_0 is a fixed point. Otherwise, we have

$$0 < d(x_0, f(x_0)) \le d(x, f(x_0)) \quad \text{for all } x \in X.$$

In this case, we show that $x_0 \in \mathrm{Bd}\, X$.
Suppose that $x_0 \in \mathrm{Int}\, X$. Then there exists an $r > 0$ such that

$$B(x_0, r) \subset X \quad \text{and} \quad r < d(x_0, f(x_0)) \le d(y, f(x_0)) \quad \text{for all } y \in B(x_0, r).$$

Then there is a $y_0 \in B(x_0, r) \cap B(f(x_0), d(x_0, f(x_0)) - r)$ by the hyperconvexity.
Hence,

$$d(y_0, f(x_0)) \leq d(x_0, f(x_0)) - r < d(x_0, f(x_0)),$$

which is a contradiction. Therefore, $x_0 \in \mathrm{Bd}\, X$.
This completes our proof. □

Remark. This is a refined version of Khamsi [5, Lemma]. We modified Khamsi's proof where KKM Theorem was exploited instead of Theorem 1.

From Theorem 4, we have the following fixed point theorems:

Theorem 5. *Let H be a hyperconvex space, $X \in \mathscr{A}(H)$ compact, and $f \in \mathbb{C}(X, H)$. Then f has a fixed point if one of the following conditions holds for all $x \in \mathrm{Bd}\, X$ such that $x \neq f(x)$:*
(i) *There exists a $y \in X$ such that*

$$d(x, f(x)) > d(y, f(x)).$$

(ii) *There exists a $\beta \in (0, 1)$ such that*

$$X \cap B(f(x), \beta d(x, f(x)) \neq \emptyset.$$

(iii) *There exists an $\alpha \in (0, 1)$ such that*

$$X \cap B(x, \alpha d(x, f(x))) \cap B(f(x), (1 - \alpha) d(x, f(x))) \neq \emptyset.$$

(iv) $f(x) \in X$.

Proof. (i) Suppose that f has no fixed point. Then by Theorem 4, there exists an $x_0 \in \mathrm{Bd}\, X$ such that

$$0 < d(x_0, f(x_0)) \leq d(y, f(x_0)) \quad \text{for all } y \in X.$$

This contradicts condition (i).
(ii) For any $x \in X$ such that $x \neq f(x)$, there exists a $y \in X$ such that

$$y \in X \cap B(f(x), \beta d(x, f(x)).$$

Then

$$d(y, f(x) \leq \beta d(x, f(x)) < d(x, f(x)).$$

Therefore (ii) implies (i).
(iii) Clearly (iii) implies (ii).
(iv) Clearly (iv) implies (ii).

Remark. Case (iii) is due to Khamsi [5, Theorem 6], and a particular form of case (iv) to Espinola-Garcia [3, Lemma 1], by different method. Note that case (iv) is analogous to theorems of Brouwer, Schauder and Rothe.

The following is known as the Schauder conjecture which is almost seventy years old:

Problem. *Let X be a compact convex subset of a (metrizable) topological vector space. Does any $f \in \mathbb{C}(X,X)$ have a fixed point?*

This is not solved yet. Note that if X is metrizable and every ball is convex, then the problem is affirmative. See [9, Corollary 13.2]. However, if X is a compact admissible subset of a hyperconvex space, then any $f \in \mathbb{C}(X,X)$ has a fixed point by Theorem 5(iv).

References

[1] N. Aronszajn, P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, Pacific J. Math. 6 (1956) 405–439.
[2] J.B. Baillon, Nonexpansive mappings and hyperconvex spaces, Contemp. Math. 72 (1988) 11–19.
[3] R. Espinola-Garcia, Darbo-Sadovski's theorem in hyperconvex metric spaces, Rend. Circolo Mat. Palermo, Serie II, Supp. 40 (1996) 129–137.
[4] E. Jawhari, D. Misane, M. Pouzet, Retracts: graphs and ordered sets from the metric point of view, Contemp. Math. 57 (1986) 175–226.
[5] M.A. Khamsi, KKM and Ky Fan theorems in hyperconvex metric spaces, J. Math. Anal. Appl. 204 (1996) 298–306.
[6] M.A. Khamsi, M. Lin, R.C. Sine, On the fixed points of commuting nonexpansive maps in hyperconvex spaces, J. Math. Anal. Appl. 168 (1992) 372–380.
[7] M.A. Khamsi, S. Reich, Nonexpansive mappings and semigroups in hyperconvex spaces, Math. Japonica 35 (1990) 467–471.
[8] M. Lin, R.C. Sine, Retractions on the fixed point set of semigroups of nonexpansive maps in hyperconvex spaces, Nonlinear Anal. TMA 15 (1990) 943–954.
[9] S. Park, Generalizations of Ky Fan's matching theorems and their applications, J. Math. Anal. Appl. 141 (1989) 164–176.
[10] S. Park, Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps, J. Korean Math. Soc. 31 (1994) 493–519.
[11] S. Park, H. Kim, Coincidence theorems of admissible maps on generalized convex spaces, J. Math. Anal. Appl. 197 (1996) 173–187.
[12] S. Park, H. Kim, Foundations of the KKM theory on generalized convex spaces, J. Math. Anal. Appl. 209 (1997) 551–571.
[13] R.C. Sine, On linear contraction semigroups in sup norm spaces, Nonlinear Anal. TMA 3 (1979) 885–890.
[14] R.C. Sine, Hyperconvexity and approximate fixed points, Nonlinear Anal. TMA 13 (1989) 863–869.
[15] R.C. Sine, Hyperconvexity and nonexpansive multifunctions, Trans. Amer. Math. Soc. 315 (1989) 755–767.
[16] P. Soardi, Existence of fixed points of nonexpansive mappings in certain Banach lattices, Proc. Amer. Math. Soc. 73 (1979) 25–29.