GENERALIZED BIRKHOFF-KELLOGG TYPE THEOREMS AND APPLICATIONS

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Abstract. We obtain very general Birkhoff-Kellogg type theorems on eigenvectors of broad classes of compact multimaps which appear in nonlinear analysis and algebraic topology. These are applied to fixed point and best approximation problems on such classes of multimaps defined on spheres of normed vector spaces.

1. Introduction

We obtain very general Birkhoff-Kellogg type theorems on eigenvectors as simple consequences of fixed point theorems. In fact, we obtain generalized forms of such theorems on broad classes of compact multimaps and apply them to fixed point and best approximation problems on spheres of normed vector spaces.

In 1922, Birkhoff and Kellogg [BK] obtained a result on invariant directions of continuous maps defined on function spaces. One of its generalizations (see Granas *et al.* [GGJ, Gr1]) is as follows:

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Theorem. *Let E be an infinite dimensional Banach space and S the boundary of the unit ball $B = B(0,1)$. Let $h : S \to E$ be a compact map such that $\|h(x)\| \geq \alpha > 0$ for all $x \in S$. Then there exist $x \in S$ and $\lambda > 0$ such that $x = \lambda h(x)$; that is, h has an invariant direction.*

This has been generalized by a number of authors. See [Y, FuM, M, CI, DG, MV, CY, FoM, P6] and others. For an overview and up to date developments on eigenvector problems, see [FoM].

On the other hand, Nussbaum [N] showed that any k-set-contraction $f : S \to S$, $k < 1$, has a fixed point, and Massatt [Ms] extended this to a condensing map $f : S \to S$. Note that any compact map in a Banach space is condensing. Later Lin [Li2] extended and applied Massatt's result to best approximation and fixed point problems for condensing non-selfmaps defined on spheres. Lin's results were strengthend and extended by the present author [P6]. Note that most of the above-mentioned works concern with single-valued continuous maps and were obtained from degree theory.

Recently, the present author [P2-7] initiated the study of fixed points of broad classes of multimaps called admissible (see section 2) and obtained very general fixed point theorems and some related results.

In this paper, we obtain the Birkhoff-Kellogg type theorems for admissible compact multimaps from a fixed point theorem in [P7] and apply them to fixed point and best approximation problems for such multimaps defined on spheres of normed vector spaces of infinite dimension.

2. Preliminaries

A *multimap* or *set-valued map* (simply, *map*) $F : X \to 2^Y$ is a function with nonempty set-values $F(x) \subset Y$ for each $x \in X$. The set $\{(x, y) : y \in F(x)\}$ is called either the *graph* of F or, simply, F. So $(x, y) \in F$ if and only if $y \in F(x)$.

For topological spaces X and Y, a map $F : X \to 2^Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X. It is well-known that if Y is compact Hausdorff and $F(x)$ is closed for each $x \in X$, then F is u.s.c. if and only if the graph of F is closed in $X \times Y$. A map $F : X \to 2^Y$ is said to be *compact* if $F(X)$ is contained in a compact subset of Y. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. In a topological vector space, any convex hulls of its finite subsets will be called *polytopes.*

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of all maps $F : X \to 2^Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of all finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values) [Gr], the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (with values consisting of one or more m acyclic components, where m is fixed) [Gr], the approachable maps $\mathbb{A}$ in topological vector spaces [BD], admissible maps in the sense of Górniewicz [G], permissible maps in Dzedzej [Dz], and others. Moreover, we define

$F \in \mathfrak{A}_c^\sigma(X, Y) \iff$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

For examples of admissible classes of multimaps, see [P2-8], [PK]. Recently, the author established the KKM theory and the fixed point theory for admissible maps; see [P3-6].

In this paper, we assume that $\mathfrak{A}$ satisfies the following:

(∗) if $F \in \mathfrak{A}(X, E)$, where E is a topological vector space and $X \subset E$, and if $\lambda > 0$, then $\lambda F \in \mathfrak{A}(X, E)$ where $(\lambda F)(x) := \lambda(F(x)) \subset E$ for $x \in X$.

A nonempty subset of X of a Hausdorff topological vector space E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exist a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. For examples of admissible sets, see [P8] and references therein.

We need the following due to the author [P8].

Theorem 0. *Let X be an admissible convex subset of a Hausdorff topological vector space E. If $T \in \mathfrak{A}_c^\kappa(X, X)$ is compact, then T has a fixed point.*

Note that Theorem 0 has a large number of historically well-known particular forms. See [P8].

Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, respectively.

3. The Birkhoff-Kellogg type theorems

For a subset X of a vector space E and a multifunction $F : X \to 2^E$, we say that F has *an eigenvalue* (*a proper value*) if the inclusion

$$\mu x \in Fx$$

has a solution $x_0 \in X$ for some real $\mu \neq 0$, and that F has *an invariant direction* (*a positive eigenvalue*) whenever $\mu > 0$.

From Theorem 0, we obtain the following generalized Birkhoff-Kellogg type theorems :

Theorem 1. *Let U be a convex neighborhood of 0 in a Hausdorff topological vector space E such that $\overline{U}$ is admissible, and $F \in \mathfrak{A}_c^\kappa(\operatorname{Bd} U, E)$ a compact map. Suppose that there is a compact extension $G \in \mathfrak{A}_c^\kappa(\overline{U}, E)$ of F such that*

(0) $\lambda G(\overline{U}) \cap \overline{U} = \emptyset$ *for some number λ.*

Then F has at least an eigenvalue.

Proof. Note that $\lambda G \in \mathfrak{A}_c^\kappa(\overline{U}, E)$ is compact and has no fixed point. Let $p : E \to \mathbb{R}$ be the Minkowski functional of U. Since $0 \in U$, p is continuous. Define $r : E \to \overline{U}$ by $r(x) = x$ for $x \in \overline{U}$ and $r(x) = p(x)^{-1}x$ for $x \notin \overline{U}$. Then r is a continuous retraction of E onto $\overline{U}$. Let $F' = r(\lambda G) \in \mathfrak{A}_c^\kappa(\overline{U}, \overline{U})$. Since λG is compact, so is F'. Therefore, by Theorem 0, F' has a fixed point $x_0 \in \overline{U}$; that is, $x_0 \in r(\lambda G)(x_0)$. We have $x_0 = r(y_0)$ for some $y_0 \in (\lambda G)(x_0)$. Note that $y_0 \notin \overline{U}$ by (0). Therefore, $x_0 = r(y_0) = p(y_0)^{-1}y_0 \in \operatorname{Bd} U$ and hence $p(y_0)x_0 = y_0 \in (\lambda G)(x_0) = (\lambda F)(x_0)$. This completes our proof.

Remark. If $\lambda > 0$, then F has an invariant direction.

Theorem 2. *Let $U, E,$ and F be the same as in Theorem 1. Suppose that*

(1) $\operatorname{Bd} U$ *is a retract of $\overline{U}$; and*

(2) $\lambda F(\operatorname{Bd} U) \cap \overline{U} = \emptyset$ *for some number λ.*

Then F has an eigenvalue.

Proof. Let $r' : \overline{U} \to \operatorname{Bd} U$ be the retraction and $G = Fr' \in \mathfrak{A}_c^\kappa(\overline{U}, E)$. Then G is a compact extension of F and (2) implies (0). Applying Theorem 1, we have the conclusion.

Remarks. 1. If $F = f \in \mathbb{C}(\overline{U}, E)$, Theorem 2 reduces to Yamamuro [Y, Theorem 2], which extends the Birkhoff-Kellogg theorem.

2. A slightly different version of Theorem 2 was due to Park [P6, Theorem 8] with different proof using a Leray-Schauder type principle.

From Theorem 2, we obtain the following:

Theorem 3. *Let S be the unit sphere of a normed vector space E of infinite dimension, and $F \in \mathfrak{A}_c^\kappa(S, E)$ a compact map such that $0 \notin \overline{F(S)}$. Then F has an invariant direction.*

Proof. Let B be the unit ball. Since E is infinite dimensional, by Dugundji [D, Lemma 6.1], $S = \operatorname{Bd} B$ is a retract of B. Since $0 \notin \overline{F(S)}$ and $\overline{F(S)}$ is compact, there exists a number $\lambda > 1$ such that $\lambda F(S) \cap B = \emptyset$. Therefore, by Theorem 2, F has an invariant direction.

Remarks. 1. Furi and Martelli [FuM, Theorem 6; M, Theorem 2] is a Banach space version of Theorem 3 for $F \in \mathbb{V}(S, E)$.

2. Even for a single-valued map $F = f \in \mathbb{C}(S, E)$, Theorem 3 improves the Birkhoff-Kellogg theorem.

3. Martelli [M] gave an example that Theorem 3 does not hold if the compact map F is replaced by a single-valued condensing map or a k-set-contraction, $k < 1$.

By a *positive cone* in a normed vector space E we mean a closed convex subset C of E such that $C \cap (-C) = \{0\}$, $\alpha C = C$ for every $\alpha > 0$, and C has nonzero vectors.

Theorem 4. *Let S be the unit sphere and C a positive cone of a normed vector space E of infinite dimension. Let $F \in \mathfrak{A}_c^\kappa(S \cap C, C)$ be a compact map such that $0 \notin \overline{F(S \cap C)}$. Then F has an invariant direction.*

Proof. The set $S \cap C$ is an absolute retract since it is a retract of the convex set $C \backslash \{0\}$ (See Hu [H]). Therefore, there exists a retraction $r : S \to S \cap C$. Consider the map $Fr \in \mathfrak{A}_c^\kappa(S, C) \subset \mathfrak{A}_c^\kappa(S, E)$ and apply Theorem 3.

Remarks. 1. For $F \in \mathbb{V}(S \cap C, C)$, Theorem 4 reduces to a form of Furi and Martelli [FM, Theorem 7].

2. Note that Theorem 4 simplifies results due to Morgenstern and Schaefer. See Bonsall [B, pp.51-52].

3. For a single-valued map F, a similar result to Theorem 4 was proved in [KL, GGJ, Gr1] by different arguments.

4. Fixed point and best approximation theorems on spheres

In this section, we obtain some consequences of Theorems 3 and 4. The following is immediate from Theorem 4:

Theorem 5. *Let S be the unit sphere and C a positive cone of a normed vector space E of infinite dimension, and $F \in \mathfrak{A}_c^\kappa(S \cap C, S \cap C)$ a compact map. Then F has a fixed point.*

Remark. Fournier and Martelli [FoM, Corollary 1.5] obtained a particular form of Theorem 5 for a Banach space E and $F \in \mathbb{V}_c(S \cap C, S \cap C)$, by using index theory.

From Theorem 3, we have

Theorem 6. *Let S be the sphere with center 0 and radius $a > 0$ in a normed vector space E of infinite dimension. Then any compact map $F \in \mathfrak{A}_c^\kappa(S, S)$ has a fixed point.*

Remark. Fournier and Martelli [FoM, Corollary 1.3] obtained a particular form of Theorem 6 for a Banach space E and $F \in \mathbb{V}_c(S, S)$, by using index theory. Note that, for a Banach space E, Massatt [Ms, Theorem 3] showed that the compact map F in Theorem 6 can be replaced by a single-valued condensing map, and Fournier and Martelli [FoM, Corollary 3.7] by an α-contraction $F \in \mathbb{V}_c(S, S)$ with constant $p < 1$.

Let X be a subset of a vector space E and $x \in E$. The *inward set* $I_X(x)$ of X at x is defined by

$$I_X(x) = \{x + r(y - x) : y \in X,\ r > 0\}.$$

For a normed vector space E,

$$d(x, X) = \inf\{\|x - y\| : y \in X\}.$$

From Theorem 6 we obtain the following best approximation theorem:

Theorem 7. *Let S be the sphere with center 0 and radius $a > 0$ in a normed vector space E of infinite dimension, and $F \in \mathfrak{A}_c^\sigma(S, E)$ a compact map such that $\|y\| \geq a$ for all $y \in F(S)$. Then either F has a fixed point $u \in S$ or there exist a point $u \in S$ and a point $v \in F(u)$ such that*

$$0 < \|u - v\| = d(v, \overline{I}_B(u)).$$

Proof. Let B be the closed ball with center 0 and radius a, and $r : E\backslash \text{Int}\, B \to S$ the radial projection; that is, $r(x) = x$ for $x \in S$ and $r(x) = ax/\|x\|$ for $x \notin S$. Then r is a continuous retraction and $rF \in \mathfrak{A}_c^\kappa(S, S)$ is compact. Therefore, by Theorem 6, rF has a fixed point $u \in S$; that is, $u \in rF(u)$. Hence $u = r(v)$ for some $v \in F(u)$. If $v \in S$, then $u = r(v) = v \in F(u)$ and u is a fixed point. If $v \notin S$, then

$$0 < \|u - v\| = \|r(v) - v\| = \|\frac{a}{\|v\|}v - v\| = \|v\| - a.$$

For any $x \in B$, we have

$$\|v\| - a \leq \|v\| - \|x\| \leq \|v - x\|$$

and hence

$$0 < \|u - v\| = d(v, B).$$

Now, we show that

$$\|u - v\| \leq \|v - x\| \quad \text{for all } x \in I_B(u).$$

In fact, for $x \in I_B(u)\backslash B$, there exist $y \in B$ and $c > 1$ such that $x = u + c(y - u)$. Suppose that $\|u - v\| > \|v - x\|$. Since

$$\frac{1}{c}x + (1 - \frac{1}{c})u = y \in B,$$

we have

$$\|v-y\| \le \frac{1}{c}\|v-x\| + (1-\frac{1}{c})\|v-u\| < \|u-v\|,$$

which contradicts $\|u-v\| = d(v,B)$. Moreover, since $\|\ \|$ is continuous, we have

$$\|u-v\| \le \|v-x\| \quad \text{for all } x \in \overline{I}_B(u).$$

This completes our proof.

Remarks. 1. If E is a Banach space and F is a single-valued condensing map, then the conclusion still holds. See Lin [Li, Theorem 1] and Park [P7, Theorem 2].

2. Without the compactness of F, the conclusion of Theorem 7 does not hold. For example, the antipodal map on S is not compact. Moreover, see the example in [Li1].

From Theorem 7, we have the following fixed point theorems which extend Theorem 6:

Theorem 8. *Let $S, E,$ and F be the same as in Theorem 7 such that $\|y\| \ge a$ for all $y \in F(S)$. Then F has a fixed point if, for each $x \in S\backslash F(x)$, one of the following conditions holds:*

(i) *For each $y \in F(x)$, there exists a $z \in \overline{I}_B(x)$ such that*

$$\|x-y\| > \|y-z\|.$$

(ii) *For each $y \in F(x)$, there exists a number λ (real or complex, depending on whether E is real or complex) such that*

$$|\lambda| < 1 \quad \textit{and} \quad \lambda x + (1-\lambda)y \in \overline{I}_B(x).$$

(iii) $F(x) \subset \overline{I}_B(x)$.

(iv) *For each $y \in F(x)$, there exists a $z \in B$ such that*

$$\|x-y\| > \|y-z\|.$$

(v) $\lim_{h\to 0^+} d[(1-h)x+hy,\ B]/h = 0$.

(vi) *For each $y \in F(x)$, there exists a λ (as above) such that*

$$|\lambda| < 1 \quad \textit{and} \quad \lambda x + (1-\lambda)y \in B.$$

(vii) $F(x) \subset IF_B(x) = \{x + c(u-x) : u \in B,\ \mathrm{Re}(c) > \frac{1}{2}\}$.

(viii) $F(x) \subset S$.

Proof. (i) Suppose that F has no fixed point. Then, by Theorem 7, there exist a $u \in S$ and a $v \in F(u)$ such that

$$0 < \|u - v\| = d(v, \overline{I}_B(u)).$$

By (i), for $v \in F(u)$, there exists a $z \in \overline{I}_B(u)$ such that

$$\|u - v\| > \|v - z\|.$$

This is a contradiction.

(ii) Let $z = \lambda x + (1 - \lambda)y$. If $x \neq y$, then

$$\|y - z\| = \|\lambda x - \lambda y\| = |\lambda| \, \|x - y\| < \|x - y\|$$

since $|\lambda| < 1$. Therefore, (ii) $\Longrightarrow$ (i).

(iii) For $\lambda = 0$, $\lambda x + (1 - \lambda)y = y \in \overline{I}_B(x)$ for each $y \in F(x)$. Hence, (iii) $\Longrightarrow$ (ii).

(iv) Since $z \in B \subset \overline{I}_B(x)$, (iv) $\Longrightarrow$ (i).

(v) It is known that (iii) $\Longleftrightarrow$ (v). See [P1] for references.

(vi) It is clear that (vi) $\Longrightarrow$ (ii).

(vii) It is well-known that (vii) $\Longleftrightarrow$ (vi). See [P1].

(viii) Note that (viii) implies any of (i)-(vii).

Theorem 9. *Let S, E, and F be the same as in Theorem 7 such that $\|y\| \geq a$ for all $y \in F(S)$. Then F has a fixed point if one of the following conditions holds.*

(ix) $F(x) \cap \{\alpha x : \alpha > 1\} = \emptyset$ *for each* $x \in S$.

(x) $\|x - y\|^2 \geq \|y\|^2 - a^2$ *for each* $x \in S$ *and* $y \in F(x)$.

(xi) $\|x - y\| \geq \|y\|$ *for each* $x \in S$ *and* $y \in F(x)$, $x \neq y$.

Proof. (ix) Let $r : E \backslash \operatorname{Int} B \to S$ be the radial projection. Then $rF \in \mathfrak{A}_c^\kappa(S,S)$ is a compact map. From Theorem 6, rF has a fixed point $u \in S$; that is, there exists a $u \in S$ such that $u \in rF(u)$. Hence $u = r(w)$ for some $w \in F(u)$. Then

$$u = r(w) = \frac{a}{\|w\|} w \quad \text{and} \quad w = \frac{\|w\|}{a} u \in F(u).$$

By (ix), we have $\|w\|/a \le 1$ and hence $\|w\| \le a$. On the other hand, since $\|y\| \ge a$ for all $y \in F(S)$, we have $\|w\| = a$, whence $u = w$ is a fixed point of F.

(x) If $y = \alpha x$ in $\|x - y\|^2 \ge \|y\|^2 - a^2$, then $\alpha \le 1$. Therefore, (x) implies (ix).

(xi) Clearly (xi) $\Longrightarrow$ (x) and (xi) $\Longrightarrow$ (iv) with $z = 0$.

Remarks. 1. Single-valued versions of Theorems 8 and 9 for a Banach space and condensing maps are given in [Li2, P7].

2. It is open that whether Theorems 5-9 hold or not for condensing maps F in $\mathfrak{A}_c^\kappa$.

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