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THE LERAY-SCHAUDER PRINCIPLES FOR CONDENSING APPROXIMABLE AND OTHER MULTIMAPS

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Abstract. From fixed point theorems recently due to the first author, we establish the Leray-Schauder principles for condensing multimaps which are continuous, acyclic, or approximable. We apply these principles to the Schaefer type theorems, the Birkhoff-Kellogg type theorems, fixed point theorems for non-selfmaps, and a solution of a quasi-equilibrium problem.

0. Introduction

The study of acyclic multimaps or other non-convex valued upper semicontinuous multimaps in algebraic topology has long history. One

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of the remarkable contributions to this field is O'Neill's work [O] on continuous multimaps. On the other hand, recently, Ben-El-Mechaiekh *et al.* [BD1-3, BI, BC] have studied other types of non-convex valued multimaps called approachable or approximable.

In [BI], Ben-El-Mechaiekh and Idzik gave a proof of a Leray-Schauder type theorem for compact approximable multimaps using a matching theorem of Ky Fan on closed covers of convex sets. Subsequently, the present author [P5] gave a Schöneberg type proof of the theorem of [BI] and applied it to obtain a Schaefer type theorem, the Birkhoff-Kellogg type theorems, a Penot type theorem for non-selfmaps, and quasi-variational inequalities, all related to compact closed approximable multimaps.

In the present paper, we are concerned with *acyclic* multimaps, *O'Neill* maps, and *approximable* multimaps defined on a class of closed convex subsets of topological vector spaces not necessarily locally convex. Our aim is to establish the Leray-Schauder type principles from fixed point theorems for condensing multimaps and to extend the results in [P5] to noncompact cases. We apply these principles to the Schaefer type theorems, the Birkhoff-Kellogg type theorems, fixed point theorems for non-selfmaps, and a quasi-equilibrium problem, all related to condensing multimaps of the above mentioned type.

Our arguments are based on a general Schauder type fixed point theorem for compact multimaps in the class $\mathfrak{B}$ of "better" admissible maps (see Park [P6,7,9]) defined on admissible convex subsets of a topological vector space.

1. Preliminaries

A t.v.s. means a Hausdorff topological vector space. Int, Bd, $\overline{\quad}$, and co denote the interior, boundary, closure, and convex hull, respectively.

For subsets X and Y of t.v.s. E and F, respectively, a *multimap* or, simply a *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty compact values. Note that $y \in Tx$ is equivalent to $x \in T^{-}y$ and, for $B \subset Y$, $T^{-}(B) := \{x \in X : Tx \cap B \neq \emptyset\}$. For $U \subset X$, $\mathrm{Bd}_X U$ denote the boundary of U in X.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if $T^{-}(B)$ is closed for each closed $B \subset Y$; *lower semicontinuous* (l.s.c.)

if $T^-(B)$ is open for each open $B \subset Y$; *continuous* if it is u.s.c. and l.s.c.; *closed* if it has the closed graph $\mathrm{Gr}(T) \subset X \times Y$; and *compact* if its range $T(X)$ is contained in a compact subset of Y.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. A map $T : X \multimap Y$ is said to be *acyclic* if it is u.s.c. with acyclic values; and called an *O'Neill map* [O] if it is continuous and has values of one or m acyclic components, where m is fixed.

Given two open neighborhoods U and V of the origins in E and F, respectively, a (U, V)-*approximative continuous selection* of $T : X \multimap Y$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (T[(x + U) \cap X] + V) \cap Y \text{ for every } x \in X.$$

A map $T : X \multimap Y$ is said to be *approximable* if its restriction $T|_K$ to any compact subset K of X admits a (U, V)-approximative continuous selection for every U and V as above; see [BI].

For examples of approximable maps, see [BD1-3, BI, BC].

Let E be a t.v.s. and C a lattice with a least element, which is denoted by 0. A function $\Phi : 2^E \to C$ is called a *measure of noncompactness* on E provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ if and only if X is relatively compact;
(2) $\Phi(\overline{\mathrm{co}}\, X) = \Phi(X)$; and
(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

It follows that $X \subset Y$ implies $\Phi(X) \leq \Phi(Y)$.

The above notion is a generalization of the set-measure γ and the ball-measure χ of noncompactness defined either in terms of a family of seminorms or a norm. For details, see [PF1,2].

For $D \subset E$, a map $T : D \multimap E$ is said to be Φ-*condensing* provided that if $X \subset D$ and $\Phi(X) \leq \Phi(T(X))$, then X is relatively compact; that is, $\Phi(X) = 0$.

From now on, we assume that Φ is a measure of noncompactness on the given t.v.s. E if necessary.

Note that any map defined on a compact set or any compact map is $\Phi-$condensing. Especially, if E is locally convex, then a compact map $T : D \multimap E$ is γ- or χ-condensing whenever D is complete or E is quasi-complete.

2. A new fixed point theorem

In this section, we introduce a general Schauder type fixed point theorem on admissible maps defined on admissible convex subsets of a t.v.s..

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [K]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible; see Nagumo [N]. Other examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, and others. Moreover, any locally convex subset of an F-normable t.v.s. and any compact convex locally convex subset of a t.v.s. is admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space l^2 is known. For details, see Klee [K], Hadžić [H], Weber [W1,2], and references therein.

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is any convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

We define the *"better" admissible class* $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \Longleftrightarrow F : X \multimap Y$ is a map such that for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P) : P \multimap P$ has a fixed point.

The following fixed point theorem was due to recently by the author [P7,9]:

Theorem A. *Let E be a t.v.s. and X an admissible convex subset of E. Then any closed compact map $T \in \mathfrak{B}(X,X)$ has a fixed point.*

In [P7,9], it was shown that our new theorem subsumes more than sixty known or possible particular forms, and generalizes them in terms of the involving spaces and the multimaps as well.

In this paper, an *AOA map* means any of acyclic maps, O'Neill maps, or u.s.c. approximable maps. It is known that any AOA map belongs to the class $\mathfrak{B}$. For other examples of maps in $\mathfrak{B}$, see [P6,7,9].

Our basic tool is the following:

THEOREM B. *Let X be an admissible convex subset of a t.v.s. E and $T : X \multimap X$ an AOA map. If T is compact, then T has a fixed point.*

In case E is locally convex, Theorem B was first due to Park [P1-3, PSW] for acyclic maps, to Park [P4,5] for approximable maps, and Theorem B for O'Neill maps was recently obtained by Park [P6].

3. The Leray-Schauder type principles

In this section, we deduce a Leray-Schauder principle by the method of Schöneberg [Sö]. From the main result, we obtain Leray-Schauder type fixed point theorems or the Leray-Schauder alternatives.

The following is known by many authors; for example, see Mehta, Tan, and Yuan [MTY, Lemma 1].

LEMMA. *Let D be a nonempty closed convex subset of a t.v.s. E and $T : D \multimap D$ a Φ-condensing map. Then there exists a nonempty compact convex subset K of D such that $T(K) \subset K$.*

Note that even if D is admissible, we can not say that K is admissible in E. Therefore, we need the following concept:

A nonempty subset D of a t.v.s. E is said to be q-*admissible* if any nonempty compact convex subset K of D is admissible.

We give some examples of q-admissible sets as follows:

(1) Any nonempty locally convex subset of a t.v.s.

(2) Any nonempty subset of a locally convex t.v.s.

(3) Any nonempty subset of a t.v.s. E on which its topological dual E^* separates point. Note that any compact convex subset of such a space E is affinely embeddable in a locally convex t.v.s.; see [W2].

It should be noted that an admissible t.v.s. (in the sense of Klee [K]) and a q-admissible t.v.s. can be also defined.

From Theorem B and Lemma, we have the following:

THEOREM 1. *Let D be a q-admissible closed convex subset of a t.v.s. E and $T : D \multimap D$ a Φ-condensing AOA map. Then T has a fixed point.*

Proof. By Lemma, there is a compact convex subset K of D such that $T(K) \subset K$. Since D is q-admissible, K is admissible. Moreover, $T|_K$ is an AOA map and compact, and hence has a fixed point by Theorem B.

From Theorem 1, we deduce the following Leray-Schauder type principle by the method of Schöneberg [Sö]:

THEOREM 2. *Let X be a q-admissible closed convex subset of a t.v.s. E such that $0 \in X$, $U \subset X$ an open neighborhood of 0 (in X), and $H : [0,1] \times \overline{U} \multimap X$ an AOA map. Suppose that*
(1) *$x \notin H(t,x)$ for $t \in [0,1]$ and $x \in \mathrm{Bd}_X U$;*
(2) *$\lambda x \notin H(1,x)$ for $\lambda > 1$ and $x \in \mathrm{Bd}_X U$; and*
(3) *$Q \subset \overline{U}$ and $\Phi(Q) \le \Phi(H([0,1] \times Q))$ imply that Q is relatively compact.*

Then there exists an $x \in \overline{U}$ such that $x \in H(0,x)$.

Proof. Let $R \subset X$ be defined by

$$R := \{x \in \overline{U} : x \in H(t,x) \text{ for some } t \in [0,1]\} \\ \cup \{x \in \overline{U} : x \in tH(1,x) \text{ for some } t \in [0,1]\}.$$

Then $R \neq \emptyset$ (since $0 \in R$) and R is compact (since H is u.s.c. and $R \subset \overline{\mathrm{co}}(\{0\} \cup H([0,1] \times R))$. Note that $R \cap \mathrm{Bd}_X U = \emptyset$ (by (1) and (2)). Since X is completely regular, there exists a continuous function $r : X \to [0,1]$ such that $r(x) = 0$ for $x \in R$ and $r(x) = 1$ for $x \in \mathrm{Bd}_X U$. Now define a map $G : X \multimap X$ by

$$Gx := \begin{cases} H(2r(x), x), & r(x) \le \frac{1}{2} \text{ and } x \in \overline{U}, \\ 2(1 - r(x))H(1,x), & r(x) \ge \frac{1}{2} \text{ and } x \in \overline{U}, \\ \{0\}, & x \notin \overline{U}. \end{cases}$$

If H is u.s.c., then G is u.s.c. since r is continuous and $r(x) = 1$ for $x \in \mathrm{Bd}_X U$. Moreover, if H is acyclic, so is G. Further, if H is an O'Neill map then so is G.

We show that if H is approximable, then G is also approximable. In fact, let K be a compact subset of X and V_1, V_2 two open convex neighborhoods of 0 in E. Note that, for any $\varepsilon > 0$, $V_1' := (-\varepsilon, \varepsilon) \times V_1$ is an open convex neighborhood of $\mathbb{R} \times E$. Since H is approximable, there exists a (V_1', V_2)-approximate continuous selection $h : [0,1] \times (K \cap \overline{U}) \to X$ of $H|_{[0,1]\times(K\cap\overline{U})}$ satisfying

$$h(t,x) \in (H[((t,x) + V_1{}') \cap ([0,1] \times (K \cap \overline{U}))] + V_2) \cap X$$

for $(t,x) \in [0,1] \times (K \cap \overline{U})$. Define $g : K \to X$ by

$$g(x) := \begin{cases} h(2r(x), x), & r(x) \le \frac{1}{2} \text{ and } x \in K \cap \overline{U}, \\ 2(1 - r(x))h(1,x), & r(x) \ge \frac{1}{2} \text{ and } x \in K \cap \overline{U}, \\ \{0\}, & x \notin K \cap \overline{U}. \end{cases}$$

Then g is continuous and it can be checked that

$$g(x) \in (G[(x + V_1) \cap K] + V_2) \cap X \quad \text{for} \quad x \in K.$$

This shows that G is approximable.

Moreover, G is Φ-condensing. In fact, let $Q \subset X$ such that $\Phi(Q) \le \Phi(G(Q))$. From the definition of G, we have

$$G(Q) \subset \overline{\mathrm{co}}(\{0\} \cup H([0,1] \times (Q \cap \overline{U}))).$$

If $Q \cap \overline{U} = \emptyset$, then $\Phi(Q) \le \Phi(G(Q)) \le \Phi(\{0\}) = 0$ and hence Q is relatively compact. If $Q \cap \overline{U} \neq \emptyset$, then we have

$$\begin{aligned} \Phi(Q \cap \overline{U}) &\le \Phi(Q) \le \Phi(G(Q)) \\ &\le \Phi(\overline{\mathrm{co}}(\{0\} \cup H([0,1] \times (Q \cap \overline{U})))) \\ &\le \Phi(H([0,1] \times (Q \cap \overline{U}))) \end{aligned}$$

and hence $Q \cap \overline{U}$ is relatively compact by (3). Therefore $H([0,1] \times (Q \cap \overline{U}))$ is relatively compact (for H is u.s.c. and compact-valued), whence

$$\Phi(Q) \le \Phi(H([0,1] \times (Q \cap \overline{U}))) = 0;$$

that is, Q is relatively compact.

Since G is a Φ-condensing AOA map, it has a fixed point $x_0 \in X$ by Theorem 1. Note that $x_0 \in Gx_0$ implies $x_0 \in R$ and $r(x_0) = 0$. Therefore, $x_0 \in \overline{U}$ and $x_0 \in H(0, x_0)$. This completes our proof.

The origin of Theorem 2 goes back to Leray and Schauder [LS]. Particular forms of Theorem 2 for a locally convex t.v.s. E are due to Browder [Br] and Potter [Pt] for single-valued maps, to Fitzpatrick and Petryshyn [FP1] for convex-valued maps, and to Schöneberg [Sö] for acyclic maps.

From Theorem 2, we have the following Leray-Schauder type fixed point theorem:

Theorem 3. *Let X be a q-admissible closed convex subset of a t.v.s. E such that $0 \in X$, $U \subset X$ an open neighborhood of 0 (in X), and $T : \overline{U} \multimap X$ an AOA map. If T is Φ-condensing, then either*

(1) *T has a fixed point; or*

(2) *$\lambda x \in Tx$ for some $\lambda > 1$ and $x \in \mathrm{Bd}_X U$.*

Proof. We use Theorem 2 with $H(t, x) = Tx$ for $t \in [0, 1]$ and $x \in \overline{U}$. Suppose that (2) does not hold. If there exists an $x_0 \in \mathrm{Bd}_X U$ such that $x_0 \in Tx_0$, then we have done. If there is no $x_0 \in \mathrm{Bd}_X U$ satisfying $x_0 \in Tx_0$, then all of the requirements of Theorem 2 are satisfied. Therefore, there exists an $x_0 \in \overline{U}$ such that $x_0 \in Tx_0$. This completes our proof.

There have appeared many particular forms of Theorem 3 in history even for a locally convex t.v.s. E. A few of them can be seen in Petryshyn [Pe2], Reich [R1-4], Gatica and Kirk [GK1,2], Fitzpatrick and Petryshyn [FP1,2], Martelli [Mr], and Lin [Ln].

The following is a particular form of Theorem 3:

Theorem 4. *Let D be a closed subset of a q-admissible t.v.s. E such that $0 \in \mathrm{Int}\, D$, and $T : D \multimap E$ an AOA map. If T is Φ-condensing, then*

(1) *T has a fixed point; or*

(2) *$\lambda x \in Tx$ for some $\lambda > 1$ and $x \in \mathrm{Bd}\, D$.*

Proof. Use Theorem 3 with $X = E$ and $U = \operatorname{Int} D$.

For a locally convex t.v.s. E, Theorem 4 for compact approximable maps was given first by Ben-El-Mechaiekh and Idzik [BI], and for Φ-condensing approximable maps by Ben-El-Mechaiekh, Chebbi, and Florenzano [BC].

Theorem 4 for acyclic maps includes many known results; for example, Petryshyn [Pe2], Furi and Martelli [FM], Martelli [Mr], Reich [R3,4], and Ben-El-Mechaiekh [Be]. As we noted earlier in [P5], Theorem 4 works for star-shaped or shrinkable sets D.

4. The Schaefer type theorems

From our Leray-Schauder type principle, we obtain the following Schaefer type theorem:

THEOREM 5. *Let E be a q-admissible t.v.s. and $T : E \multimap E$ an AOA map. If T is Φ-condensing, then either*
(1) *for each $\lambda \in (0,1)$, there exists an $x \in E$ such that $x \in \lambda Tx$; or*
(2) *the set $A := \{x \in E : x \in \mu Tx$ for some $\mu \in (0,1)\}$ is not bounded.*

Proof. Let $\lambda \in (0,1)$ and $S := \lambda T$ defined by $Sx = \lambda Tx$ for $x \in E$. Suppose that A is bounded. Then for any open convex neighborhood V of 0, there exists an $r > 0$ such that $U := rV \supset A$. Define $H : [0,1] \times \overline{U} \multimap E$ by

$$H(t,x) = (1-t)Sx \quad \text{for} \quad t \in [0,1] \text{ and } x \in \overline{U}.$$

Now we apply Theorem 2 with $X = E$.

(1) $x \notin H(t,x)$ for $t \in [0,1]$ and $x \in \operatorname{Bd} U$.

In fact, if $x \in H(t,x) = (1-t)Sx = \lambda(1-t)Tx$ for some $t \in [0,1)$, then $\lambda(1-t) \in (0,1)$ and $x \in A \subset U$. Since U is open, we should have $x \notin \operatorname{Bd} U$. On the other hand, if $x \in H(1,x) = \{0\}$, then $x \notin \operatorname{Bd} U$.

(2) For any $x \in \operatorname{Bd} U$,

$$H(1,x) \cap \{\mu x : \mu > 1\} = \{0\} \cap \{\mu x : \mu > 1\} \neq \emptyset.$$

(3) Suppose that $X \subset \overline{U}$ and $\Phi(X) \leq \Phi(H([0,1] \times X))$. Then

$$\begin{aligned}\Phi(X) &\leq \Phi(H([0,1] \times X)) \\ &\leq \Phi(\bigcup\{tT(X) : t \in [0,1]\}) \\ &\leq \Phi(\overline{\mathrm{co}}(\{0\} \cup T(X))) = \Phi(T(X)).\end{aligned}$$

Since T is Φ-condensing, X is relatively compact.

Note that, since T is an AOA map, so is H. Therefore, by Theorem 2, there exists an $x \in \overline{U}$ such that $x \in H(0,x) = Sx = \lambda Tx$. This completes our proof.

For a locally convex t.v.s., particular forms of Theorem 5 were due to Schaefer [Sc], Reich [R1,2], and Šeda [Se]. For a compact approximable map, Theorem 5 was due to Park [P5].

As in the proof of Theorem 5, we have the following equivalent form of Theorem 5. However, we can give another proof using Theorem 3.

Theorem 6. *Let E be a q-admissible t.v.s. and $T : E \multimap E$ an AOA map. If T is Φ-condensing, then either*
(1) *T has a fixed point; or*
(2) *the set $A = \{x \in E : x \in tTx$ for some $t \in (0,1)\}$ is not bounded.*

Proof. Suppose that A is bounded. Let D be a bounded neighborhood of 0 such that $A \subset \operatorname{Int} D$. Then no $y \in \operatorname{Bd} D$ satisfies $\lambda y \in Ty$ for any $\lambda > 1$. Therefore, by Theorem 3, T has a fixed point in $\overline{D}$.

For a locally convex t.v.s. E, particular forms of Theorem 6 were known by Martelli and Vignoli [MV], Martelli [Mr], Granas [Gr2], and Park [P5].

5. The Birkhoff-Kellogg type theorems

As an application of Theorem 4, we have the following generalization of the Birkhoff-Kellogg theorem [BK]:

THEOREM 7. *Let D be a closed subset of a q-admissible t.v.s. E such that $0 \in \operatorname{Int} D$ and $T : D \multimap E$ an AOA map such that $\lambda T(D) \cap D = \emptyset$ for some $\lambda \in \mathbb{R}$. If T is Φ-condensing, then $T|_{\operatorname{Bd} D}$ has an eigenvalue; that is, $\mu x \in Tx$ for some $\mu \neq 0$ and $x \in \operatorname{Bd} D$.*

Proof. Note that $\lambda \neq 0$ and $\lambda T : D \multimap E$ is an AOA map. Moreover, λT is Φ-condensing. Further, λT has no fixed point. Therefore, by Theorem 4, there exist $x \in \operatorname{Bd} D$ and $\mu > 1$ such that $\mu x \in \lambda T x$, whence we have $(\lambda^{-1}\mu)x \in Tx$, where $\lambda^{-1}\mu \neq 0$. This completes our proof.

REMARK. If $\lambda > 0$ in Theorem 7, then $T|_{\operatorname{Bd} D}$ has an invariant direction (a positive eigenvalue); that is, $\mu x \in Tx$ for some $\mu > 0$ and $x \in \operatorname{Bd} D$. Note that Theorem 7 also holds for a compact map T when E is locally convex; see [P5].

THEOREM 8. *Let S be the unit sphere of a normed vector space E of infinite dimension, and $T : S \multimap E$ an AOA map such that $0 \notin \overline{T(S)}$. If T is compact, then T has an invariant direction.*

Proof. Since E is infinite dimensional, by the Dugundji extension theorem, there exists a retraction $r : E \to S$ such that $r(x) = x/\|x\|$ if $\|x\| \geq 1$ and $\|r(x)\| = 1$ if $\|x\| \leq 1$. Let $T' = Tr : E \multimap E$. Then T' is also an AOA map. Moreover, T' is compact. Let B be the closed unit ball. Then $\lambda T'(B) \cap B = \emptyset$ for some $\lambda > 0$ since $T'(B) \subset \overline{T'(S)}$ and $0 \notin \overline{T'(S)}$. Therefore, by Theorem 7 for compact maps with $D = B$, $T'|_S$ has a eigenvalue. Since $\lambda > 0$, this eigenvalue is positive. This completes our proof.

Furi and Martelli [FM] and Martelli [Mr] obtained Theorem 8 for acyclic maps with the aid of the Lefschetz theory. Martelli [Mr] noted that Theorem 8 does not hold if T is condensing instead of compactness. Moreover, Theorem 8 for approximable maps is due to Park [P5].

Theorem 8 reduces immediately to the following:

Theorem 9. *Let S be the unit sphere of a normed vector space E. Then E is of infinite dimension if and only if any continuous compact map $f : S \to S$ has a fixed point.*

6. Fixed points of non-selfmaps

From Theorem 3, we have the following:

Theorem 10. *Let X be a q-admissible closed convex subset of a t.v.s. E such that $0 \in X$, $U \subset X$ an open convex neighborhood of 0 (in X), and $T : \overline{U} \multimap X$ an AOA map such that $T(\mathrm{Bd}_X U) \subset \overline{U}$. If T is Φ-condensing, then T has a fixed point.*

Proof. For each $x \in \mathrm{Bd}_X U$, $Tx \subset \overline{U}$ implies $Tx \cap \{\lambda x : \lambda > 1\} = \emptyset$ since $\overline{U}$ is convex and $0 \in U$. Now, the conclusion follows from Theorem 3.

From Theorems 1 and 3, we have the following:

Theorem 11. *Let X be a q-admissible closed convex subset of a t.v.s. E and $T : X \multimap E$ an AOA map such that $T(\mathrm{Bd}\, X) \subset X$. If T is Φ-condensing, then the fixed point set of T in X is nonempty and compact.*

Proof. If $\mathrm{Int}\, X = \emptyset$, then $X = \mathrm{Bd}\, X$ and $T : X \multimap X$ has a fixed point by Theorem 1. If $\mathrm{Int}\, X \neq \emptyset$, then we may assume $0 \in \mathrm{Int}\, X$. Now for each $x \in \mathrm{Bd}\, X$, $Tx \subset X$ implies $Tx \cap \{\lambda x : \lambda > 1\} = \emptyset$ since X is shrinkable; that is, $[0, 1)X \subset \mathrm{Int}\, X$. Therefore, by Theorem 3, T has a fixed point.

For a locally convex t.v.s. E, particular forms of Theorems 10 and 11 were due to Knaster-Kuratowski-Mazurkiewicz [KKM], Rothe [Ro], Eilenberg-Montgomery [EM], Granas [Gr1], Powers [Pw], Penot [P], and Park [P5].

7. Quasi-equilibrium problems

From Theorem 11, we have the following solution of a quasi-equilibrium problem:

THEOREM 12. *Let X be a q-admissible closed convex subset of a t.v.s. E, and $f : X \times E \to \mathbf{R}$ an u.s.c. function. Let $T : X \multimap E$ be an u.s.c. Φ-condensing map such that $T(\operatorname{Bd} X) \subset X$. Suppose that*

(i) *the function M defined on X by*

$$M(x) = \sup_{y \in T(x)} f(x,y) \quad \text{for} \quad x \in X$$

is l.s.c.; and

(ii) *the map $F : X \multimap E$ defined on X by*

$$F(x) = \{y \in T(x) : f(x,y) = M(x)\} \quad \text{for} \quad x \in X$$

is acyclic-valued or approximable or l.s.c. with values for the O'Neill maps.

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in T\hat{x} \quad \text{and} \quad f(\hat{x},\hat{x}) = M(\hat{x}).$$

Proof. Note that the marginal function M in (i) is actually continuous since f is u.s.c. and T is a compact-valued u.s.c. map, by the well-known result of Berge [Br]. Now, each $F(x)$ is nonempty and compact. Moreover, F is a closed map. In fact, let $(x_\alpha, y_\alpha) \in \operatorname{Gr}(F)$, the graph of F, and $(x_\alpha, y_\alpha) \to (x,y)$ in $X \times E$. Then

$$\begin{aligned} f(x,y) &\geq \overline{\lim_\alpha} f(x_\alpha, y_\alpha) = \overline{\lim_\alpha} M(x_\alpha) \\ &\geq \underline{\lim_\alpha} M(x_\alpha) \geq M(x) \end{aligned}$$

and, since $\operatorname{Gr}(T)$ is closed in $X \times Y$, $y_\alpha \in T(x_\alpha)$ implies $y \in T(x)$. Hence $(x,y) \in \operatorname{Gr}(F) \subset \operatorname{Gr}(T)$. Since T is u.s.c., so is F. Therefore, $F : X \multimap E$ is an AOA map satisfying $F(\operatorname{Bd} X) \subset T(\operatorname{Bd} X) \subset X$. Moreover, F is also Φ-condensing. In fact, suppose that $D \subset X$ and $\Phi(D) \leq \Phi(F(D))$. Then $\Phi(D) \leq \Phi(F(D)) \leq \Phi(T(D))$. Since T is Φ-condensing, we have $\Phi(D) = 0$ and hence F is Φ-condensing. Hence, by Theorem 11, F has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T(\hat{x})$ and $f(\hat{x},\hat{x}) = M(\hat{x})$. This completes our proof.

REMARK. If $f = 0$, then Theorem 12 reduces to Theorem 11. If E is locally convex and T is compact and approximable, Theorem 12 reduces to Park [P5, Theorem 8]. Note that, from Theorem 12, we can deduce a number of variational or variational-like inequalities as in [P5, PC, PP].

8. Corrections of earlier works

The concept of q-admissible subsets of a t.v.s. is sufficient to correct some of the earlier works of the first author [P7-9].

As in the proof of Theorem 1, we can deduce the following from Theorem A:

THEOREM C. *Let X be a q-admissible closed convex subset of a t.v.s. E. Then any closed Φ-condensing map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

This is a correct form of [P7, Theorem 3] and [P9, Corollary 2.1]. Similarly, we have to replace admissibility of X in [P9, Theorem 2 and Corollary 2.2] and [P7, Theorem 4] by q-admissibility.

Moreover, in [P8, Theorem 1], $\mathrm{cl} f(D)$ should be replaced by D. Further, each of [P8, Theorems 2-4] can be slightly improved by replacing the admissibility of K by that of D.

References

[Be] H. Ben-El-Mechaiekh, *A remark concerning a matching theorem of Ky Fan*, Chinese J. Math. **17** (1989), 309–314.

[BC] H. Ben-El-Mechaiekh, S. Chebbi, and M. Florenzano, *A Leray-Schauder type theorem for approximable maps: a simple proof*, Proc. Amer. Math. Soc. **126** (1998), 2345–2349.

[BD1] H. Ben-El-Mechaiekh and P. Deguire, *Approximation of non-convex set-valued maps*, C.R. Acad. Sci. Paris **312** (1991), 379–384.

[BD2] ——, *General fixed point theorems for non-convex set-valued maps*, C.R. Acad. Sci. Paris **312** (1991), 433–438.

[BD3] H. Ben-El-Mechaiekh and P. Deguire, *Approachability and fixed points for non-convex set-valued maps*, J. Math. Anal. Appl. **170** (1992), 477–500.

[BI] H. Ben-El-Mechaiekh and A. Idzik, *A Leray-Schauder type theorem for approximable maps*, Proc. Amer. Math. Soc. **122** (1994), 105–109.

[BK] G.D. Birkhoff and O.D. Kellogg, *Invariant points in function space*, Trans. Amer. Math. Soc. **23** (1922), 96–115.

[Br] F.E. Browder, *Problèmes Non-Linéaires*, Sém. Math. Supér., Pres. Univ. Montréal, 1966.

[EM] S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. J. Math. **68** (1946), 214–222.

[FP1] P.M. Fitzpatrick and W.V. Petryshyn, *Fixed point theorems for multivalued noncompact acyclic mappings*, Pacific J. Math. **54** (1974), 17–23.

[FP2] ———, *Fixed point theorems and the fixed point index for multivalued mappings in cones*, J. London Math. Soc. (2) **12** (1975), 75–85.

[FM] M. Furi and M. Martelli, *A degree for a class of acyclic-valued vector fields in Banach spaces*, Ann. Scuola Norm. Sup. Pisa (4) **1** (1974), 301–310 (1975).

[GK1] J.A. Gatica and W.A. Kirk, *Fixed point theorems for contraction mappings with applications to nonexpansive and pseudo-contractive mappings*, Rocky Mount. J. Math. **4** (1974), 69–79.

[GK2] ———, *A fixed point theorem for k-set-contractions defined in a cone*, Pacific J. Math. **53** (1974), 131–132.

[Gr1] A. Granas, *Theorems on antipodes and theorems on fixed points for a certain class of multi-valued mappings in Banach spaces*, Bull. Acad. Polon. Sci. **7** (1959), 271–275.

[Gr2] ———, *On the Leray-Schauder alternative*, Top. Meth. Nonlinear Anal. **2** (1993), 225–231.

[H] O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, Univ. of Novi Sad, Novi Sad, 1984.

[K] V. Klee, *Leray-Schauder theory without local convexity*, Math. Ann. **141** (1960), 286–296.

[KKM] B. Knaster, C. Kuratowski, und S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.

[LS] J. Leray and J. Schauder, *Topologie et équations fonctionnelles*, Ann. Ecole Norm. Sup. **3** (1934), 45–78.

[Ln] T.-C. Lin, *Approximation theorems and fixed point theorems in cones*, Proc. Amer. Math. Soc. **102** (1988), 502–506.

[Mr] M. Martelli, *A Rothe's type theorem for non-compact acyclic-valued maps*, Boll. Un. Mat. It. (4) **11** Suppl. Fasc. **3** (1975), 70–76.

[MV] M. Martelli and A. Vignoli, *Eigenvectors and surjectivity for α-Lipschitz mappings in Banach spaces*, Ann. Mat. Pura Appl. **94** (1972), 1–10.

[MTY] G.B. Mehta, K.-K. Tan, and X.-Z. Yuan, *Maximal elements and generalized games in locally convex topological vector spaces*, Bull. Polish Acad. Sci. Math. **42** (1994), 43–53.

[N] M. Nagumo, *Degree of mappings in convex linear topological spaces*, Amer. J. Math. **73** (1951), 497–511.

[O] R. O'Neill, *Induced homology homomorphisms for set-valued maps*, Pacific J. Math. **7** (1957), 1179–1184.

[P1] Sehie Park, *Some coincidence theorems on acyclic multifunctions and application to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, ed.). World Scientific, River Edge, NJ, 1992, pp.248–277.

[P2] ______, *Acyclic maps, minimax inequalities, and fixed points*, Nonlinear Anal. TMA **24** (1995), 1549–1554.

[P3] ______, *Fixed points of acyclic maps on topological vector spaces*, Proc. 1st World Congress of Nonlinear Analysts '92 (V. Lakshimikantham, Ed.), de Gruyter, Berlin-New York, 1996, pp.2171–2177.

[P4] ______, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc. **31** (1994), 493–519.

[P5] ______, *Fixed points of approximable maps*, Proc. Amer. Math. Soc. **124** (1996), 3109–3114.

[P6] ______, *Coincidence theorems for the better admissible multimaps and their applications*, Proc. 2nd World Congress of Nonlinear Analysts '96, Nonlinear Analysis, TMA **30** (1997), 4183–4191.

[P7] ______, *Fixed points of the better admissible multimaps*, Math. Sci. Research Hot-Line **1(9)** (1997), 1–6.

[P8] ______, *Collectively fixed points and equilibrium points of abstract economies*, Math. Sci. Research Hot-Line **1(11)** (1997), 9–13.

[P9] ______, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829.

[PC] S. Park and M.-P. Chen, *Generalized quasi-variational inequalities*, Far East J. Math. Sci. **3** (1995), 199–204.

[PP] S. Park and J.A. Park, *The Idzik type quasivariational inequalities and noncompact optimization problems*, Colloq. Math. **71** (1996), 287–297.

[PSW] S. Park, S.P. Singh, and B. Watson, *Some fixed point theorems for composites of acyclic maps*, Proc. Amer. Math. Soc. **121** (1994), 1151–1158.

[P] J.-P. Penot, *Fixed point theorems without convexity*, Bull. Soc. Math. France Mém. **60** (1979), 129–152.

[Pe1] W.V. Petryshyn, *Remarks on fixed point theorems and their extensions*, Trans. Amer. Math. Soc. **126** (1967), 43–53.

[Pe2] ______, *Structure of the fixed points set of k-set contractions*, Arch. Rational Mech. Anal. **40** (1971), 312–328.

[PF1] W.V. Petryshyn and P.M. Fitzpatrick, *A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact mappings*, Trans. Amer. Math. Soc. **194** (1974), 1–25.

[PF2] ______, *Fixed-point theorems for multivalued noncompact inward maps*, J. Math. Anal. Appl. **46** (1974), 756–767.

[Pt] A.J.B. Potter, *An elementary version of the Leray-Schauder theorem*, J. London Math. Soc. (2) **5** (1972), 414–416.

[Pw] M.J. Powers, *Lefschetz fixed point theorems for multi-valued maps*, Lect. Notes Math. **151**, Springer-Verlag, Berlin-Heidelberg-New York, 1970, pp.74–81.

[R1] S. Reich, *A fixed point theorem in locally convex spaces*, Bull. Cal. Math. Soc. **63** (1971), 199–200.

[R2] ———, *Fixed points in locally convex spaces*, Math. Z. **125** (1972), 17–31.

[R3] ———, *A remark on set-valued mappings that satisfy the Leray-Schauder conditions*, Atti Acad. Naz. Lincei **61** (1976), 193–194.

[R4] ———, *A remark on set-valued mappings that satisfy the Leray-Schauder condition.* II, Atti Acad. Naz. Lincei **66** (1979), 1–2.

[Ro] E. H. Rothe, *Zur Theorie der topologischen Ordnung und der Vektorfelder in Banachschen Räumen*, Comp. Math. **5** (1938), 177–197.

[Sc] H. Schaefer, *Über die Methode der a priori Schranken*, Math. Ann. **129** (1955), 415–416.

[Sö] R. Schöneberg, *Leray-Schauder principle for condensing multi-valued mappings in topological linear spaces*, Proc. Amer. Math. Soc. **72** (1978), 268–270.

[Se] V. Šeda, *Some fixed point theorems for multivalued mappings*, Czech. Math. J. **39** (114) (1989), 147–164.

[W1] H. Weber, *Compact convex sets in non-locally convex linear spaces, Schauder-Tychonoff fixed point theorem*, Topology, Measure, and Fractals (Warnemünde, 1991), Math. Res. **66**, Akademie-Verlag, Berlin, 1992, pp.37–40.

[W2] ———, *Compact convex sets in non-locally-convex linear spaces*, Note di Mathematica **12** (1992), 271–289.