# MINIMAX THEOREMS IN CONVEX SPACES

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**Abstract**

We apply our basic coincidence theorem in [6] to obtain new forms of minimax theorems and a saddle point result. Our results extend works of Arandjelović [1] and Sion [7], and are variants of works of Ha [3], Komornik [5], and Komiya [4].

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## 1. Introduction

Recently, in [6], we introduced a "better" admissible class $\mathfrak{B}$ of multimaps and a basic coincidence theorem for $\mathfrak{B}$ as well as a matching theorem and a KKM theorem. Those results are subsequently applied to problems related to a generalized minimax inequality in [7] and to extensions of monotone sets in [8].

In the present paper, we apply the basic coincidence theorem in [6] to obtain new forms of minimax theorems and a saddle point result.

From our basic theorem (Theorem 0), we first deduce a particular coincidence theorem (Theorem 1) extending Fan's result [2]. Then, from Theorem 1, we deduce a minimax theorem (Theorem 2), which is a variant of a result of Komornik [5] and extends a saddle point result due to Arandjelović [1] and Sion [7]. Finally, we obtain another minimax theorem (Theorem 3) involving acyclic sets, which is a variant of results of Ha [3], Komornik [5], and Komiya [4].

## 2. Coincidence Theorems

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from a set $X$ into the power set $2^Y$ of another set $Y$. Note that $y \in T(x)$ is equivalent to $x \in T^-(y)$, and $T(A) = \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

A *convex space* is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls will be called *polytopes*.

For topological spaces $X$ and $Y$, a multimap $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if, for each open subset $G$ of $Y$, the set $\{x \in X : Tx \subset G\}$ is open in $X$; and *compact* whenever $T(X)$ is relatively compact in $Y$. Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

In our previous work [6], we introduced a "better" admissible class $\mathfrak{B}$ of maps defined on a convex space $X$ as follows:

$$T \in \mathfrak{B}(X,Y) \iff \text{for any polytope } P \text{ in } X \text{ and any } f \in \mathbb{C}(F(P),P), \text{ the composite } f(T|_P) : P \multimap P \text{ has a fixed point.}$$

We give some examples of $\mathfrak{B}$ as follows:

$$t \in \mathbb{C}(X,Y) \iff t \text{ is a continuous (single-valued) function.}$$

$$T \in \mathbb{K}(X,Y) \iff T \text{ is an u.s.c. map with nonempty compact convex values, where } Y \text{ is a convex space.}$$

$$T \in \mathbb{V}(X,Y) \iff T \text{ is an acyclic map; that is, an u.s.c. map with compact acyclic values.}$$

$$T \in \Phi(X,Y) \iff T(x) \text{ is nonempty convex for each } x \in X \text{ and } T^-(y) \text{ is open for each } y \in Y, \text{ where } Y \text{ is a convex space.}$$

There are many other examples of $\mathfrak{B}$; see [6].

The following is due to the author [6, Theorem 1]:

**Theorem 0.** *Let $X$ be a convex space, $Y$ a Hausdorff space, and $T, S : X \multimap Y$ maps satisfying*

(1) $T \in \mathfrak{B}(X, Y)$ *is compact;*

(2) *for each* $y \in T(X)$, $S^{-}(y)$ *is convex; and*

(3) $\{\mathrm{Int}\ S(x) : x \in X\}$ *covers the closure* $\overline{T(X)}$.

*Then $T$ and $S$ have a coincidence point $x_0 \in X$; that is, $T(x_0) \cap S(x_0) \neq \emptyset$.*

From Theorem 0, we obtain the following theorem, which shows that there is another subclass of $\mathfrak{B}$ bigger than $\Phi$:

**Theorem 1.** *Let $X$ be a convex space, $Y$ a Hausdorff convex space, and $F, G : X \multimap Y$ maps such that*

(1.1) *$F$ is compact, $F(x)$ is convex for each $x \in X$, and* $X = \bigcup\{\mathrm{Int}\ F^{-}(y) : y \in Y\}$; *and*

(1.2) *$G^{-}(y)$ is convex for each $y \in F(X)$ and* $\overline{F(X)} = \bigcup\{\mathrm{Int}\ G(x) : x \in X\}$.

*Then $F$ and $G$ have a coincidence point.*

*Proof.* In view of Theorem 0, it suffices to show that $F \in \mathfrak{B}(X, Y)$. Let $P$ be a polytope in $X$. Since $P$ is compact, there exists a finite subset $\{y_1, y_2, \ldots, y_n\} \subset Y$ such that $P \subset \bigcup_{i=1}^{n} \mathrm{Int}\ F^{-}(y_i)$. Let $\{\lambda_i\}_{i=1}^{n}$ be the partition of unity subordinated to the cover of $P$. Define $h : P \to Y$ by

$$h(x) = \sum_{i=1}^{n} \lambda_i(x) y_i = \sum_{i \in N_x} \lambda_i(x) y_i \quad \text{for } x \in P,$$

where

$$i \in N_x \Longleftrightarrow \lambda_i(x) \neq 0 \Longrightarrow x \in \mathrm{Int}\ F^{-}(y_i) \subset F^{-}(y_i).$$

Then $y_i \in Fx$ for each $i \in N_x$. Clearly $h$ is continuous and, by (1.1), $h(x) \in \mathrm{co}\{y_i : i \in N_x\} \subset Fx$ for each $x \in P$. Therefore, $h$ is a continuous selection of $F|_P$. Since $h : P \to h(P) \subset F(P)$, for any $f : \mathbb{C}(F(P), P)$, the composite $fh : P \to P$ is a continuous selection of $f(F|_P) : P \multimap P$ and has a fixed point by the Brouwer fixed point theorem. Hence, $F \in \mathfrak{B}(X, Y)$. Now the conclusion follows from Theorem 0.

*Remark.* Theorem 1 improves a result of Fan [2].

## 3. New Minimax Theorems

In this section, $Z$ denotes a *complete linearly ordered space*; that is, a linearly ordered set whose every subset has a least upper bound. Examples are the extended real line $\overline{\mathbf{R}}$, the extended Euclidean space $\overline{\mathbf{R}}^n$, and any compact (in the Euclidean topology) subset of $\mathbf{R}^n$ with respect to the lexicographic order; see Komornik [5].

For a topological space $X$, a function $f : X \to Z$ is said to be *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] whenver $\{x \in X : f(x) > z\}$ [resp. $\{x \in X : f(x) < z\}$] is open in $X$ for each $z \in Z$.

If $X$ is compact and $f : X \to Z$ is l.s.c., then there exists an $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$. For any family $\{f_i\}_{i \in I}$ of l.s.c. functions $f_i : X \to Z$, the function $\sup_{i \in I} f_i$ is also l.s.c. See [5].

The following is the main result of this section:

**Theorem 2.** *Let $X$ be a convex space, $Y$ a Hausdorff compact convex space, and $f : X \times Y \to Z$ a function. Suppose that*

(1) *there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap (a, b) \neq \emptyset$;*

(2) *$f(x, \cdot)$ is l.s.c. on $Y$ and $\{y \in Y : f(x, y) < s\}$ is convex for each $x \in X$ and $s \in U$; and*

(3) *$f(\cdot, y)$ is u.s.c. on $X$ and $\{x \in X : f(x, y) > s\}$ is convex for each $y \in Y$ and $s \in U$.*

*Then*

$$\sup_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\sup_{x\in X} f(x,y).$$

*Proof.* Since $f(x,\cdot)$ is l.s.c., $p(x) = \min_{y\in Y} f(x,y)$ exists for each $x \in X$. Since $q(y) = \sup_{x\in X} f(x,y)$ is l.s.c. for each $y \in Y$, $q(y_0) = \min_{y\in Y} q(y)$ exists. Note that

$$p(x) = \min_{y\in Y} f(x,y) \le f(x,y) \le \sup_{x\in X} f(x,y) = q(y)$$

for all $x \in X$ and $y \in Y$. Therefore, we have

$$\sup_{x\in X} p(x) \le \min_{y\in Y} q(y).$$

Suppose that the equality does not hold. Then there exists an $s \in U$ such that

$$\sup_{x\in X} p(x) < s < \min_{y\in Y} q(y).$$

We define multimaps $T, S : X \multimap Y$ by

$$T(x) = \{y \in Y : f(x,y) < s\} \text{ and } S(x) = \{y \in Y : f(x,y) > s\}$$

for $x \in X$. Then $T(x)$ is nonempty and convex by (2), and $S(x)$ is open since $f(x,\cdot)$ is l.s.c. Moreover,

$$T^{-}(y) = \{x \in X : f(x,y) < s\} \text{ and } S^{-}(y) = \{x \in X : f(x,y) > s\}$$

for $y \in Y$. Then $T^{-1}(y)$ is open since $f(\cdot,y)$ is u.s.c., and $S^{-1}(y)$ is nonempty and convex by (3). Now, by applying Theorem 1, there exists an $x_0 \in X$ such that $T(x_0) \cap S(x_0) \neq \emptyset$. This contradicts

$$f(x_0,a) < s < f(x_0,b) \quad \text{for each } a \in T(x_0) \text{ and } b \in S(x_0).$$

This completes our proof.

*Remark.* If $U = Z$, then Theorem 2 is a consequence of Komornik [5, Theorem 2] for interval spaces with different proof.

**Corollary.** *Under the hypothesis of Theorem 2, further if $X$ is compcat, then $f$ has a saddle point.*

*Proof.* Since $f(x,\cdot)$ and $f(\cdot,y)$ are l.s.c. and u.s.c., resp., $p(x) = \min_{y\in Y} f(x,y)$ and $q(y) = \max_{x\in X} f(x,y)$ exist for each $x \in X$ and $y \in Y$. Since $p$ is u.s.c. on $X$ and $q$ is l.s.c. on $Y$, $\max_{x\in X} p(x) = p(x_0)$ and $\min_{y\in Y} q(y) = q(y_0)$ for some $x_0 \in X$ and $y_0 \in Y$. Then $(x_0, y_0)$ is a saddle point by Theorem 2. This completes our proof.

*Remark.* Corollary reduces to Arandjelović [1, Theorem 3] whenever $Z = \mathbf{R}$, which extends the Sion minimax theorem [7].

The following new minimax theorem is a variant of Theorem 2:

**Theorem 3.** *Let $X$ be a convex space, $Y$ a Hausdorff compact space, and $f : X \times Y \to Z$ a l.s.c. function such that*

(1) *there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap [a, b) \neq \emptyset$;*
(2) *for each $s \in U$ and $y \in Y$, $\{x \in X : f(x,y) > s\}$ is convex; and*
(3) *for each $s \in U$ and $x \in X$, $\{y \in X : f(x,y) \leq s\}$ is acyclic.*

*Then*

$$\sup_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \sup_{x\in X} f(x,y).$$

*Proof.* As in the proof of Theorem 2, we have

$$\sup_{x\in X} p(x) \leq \min_{y\in Y} q(y).$$

Suppose that the equality does not hold. Then there exists an $s \in U$ such that

$$\sup_{x\in X} p(x) \leq s < \min_{y\in Y} q(y).$$

We define multimaps $T, S : X \multimap Y$ by

$$T(x) = \{y \in Y : f(x, y) \leq s\} \text{ and } S(x) = \{y \in Y : f(x, y) > s\}$$

for $x \in X$. Then $T(x)$ is nonempty by the definition of $p(x)$ and closed since $f(x, \cdot)$ is l.s.c. for each $x \in X$. On the other hand, $S(x)$ is open since $f(x, \cdot)$ is l.s.c. Moreover, for each $y \in Y$,

$$S^{-}(y) = \{x \in X : f(x, y) > s\}$$

is nonempty and convex by (2).

Consider the graph of $T$

$$\mathrm{Gr}(T) = \{(x, y) \in X \times Y : f(x, y) \leq s\}.$$

Since $f$ is l.s.c., $\mathrm{Gr}(T)$ is closed in $X \times Y$. Since $Y$ is compact, $T$ is u.s.c. Note that each $T(x)$ is acyclic by (3). Hence $T$ is an acyclic map.

Therefore by Theorem 0 for $\mathbb{V}$ instead of $\mathfrak{B}$, there exists an $x_0 \in X$ such that $T(x_0) \cap S(x_0) \neq \emptyset$. This leads a contradiction as in the proof of Theorem 2.

*Remarks.* 1. In case we replace the acyclicity in (3) by convexity and if $U = Z = \mathbf{R}$, then Theorem 3 is a particular form of Ha [3, Theroem 4].

2. If we replace the acyclicity in (3) by convexity and if $U = Z$, then Theorem 3 follows from Komornik [5, Theorem 3] with different proof.

3. Komiya [4, Theorem 3] obtained a saddle point theorem whenever $U = Z = \mathbf{R}$, $f$ is continuous, and the acyclicity in (3) is replaced by convexity in Theorem 3 under an extra restriction.

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