ANOTHER FIVE EPISODES RELATED TO GENERALIZED CONVEX SPACES

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Abstract. We show that the section theorems and other results of Yuan [Y] and Wu and Xu [WX] are consequences of our works on G-convex spaces; generalized S-KKM maps due to Lin and Chang [LC] are G-KKM maps or generalized KKM maps; and Cheng's KKM theorem [C] follows from our earlier work in [PW]. Moreover, we extend Hadžić's coincidence theorem [H] of the Sehgal-Singh-Watson type. Further, we note that main results of Zhang and Zhang [ZZ] are restatements of our earlier work.

0. Introduction

This is a continuation of our previous note [P4]. Our aim in the present note is as in [P4] to show that our theory on G-convex spaces can be used to obtain improved versions of recent works of others; namely, section theorems and other results of Yuan [Y] and Wu and Xu [WX], generalized S-KKM maps due to Lin and Chang [LC], Cheng's KKM theorem in [C], and a Sehgal-Singh-Watson type theorem due to Hadžić [H]. Further, we show that main results of Zhang and Zhang [ZZ] are restatements of our earlier work.

For the terminology, we follow [P4]. However, the following are basic:

For a nonempty set A, let $\langle A\rangle$ denote the set of all nonempty finite subsets of A and $|A|$ the cardinality of A. Let Δ_n denote the standard n-simplex; that is,

$$\Delta_n = \{u \in \mathbf{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u)e_i,\ \lambda_i(u) \geq 0,\ \sum_{i=1}^{n+1} \lambda_i(u) = 1\},$$

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where e_i is the i-th unit vector in $\mathbf{R}^{n+1}$.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$.

The basic theory of G-convex spaces was studied extensively by Park and Kim [PK 1-5], which were immediately followed by works of Tan and Zhang [TZ], Tan [T], Lin and Chang [LC], Lin and Park [LP], and others.

At first, a G-convex space is defined under the extra restriction that

(0) for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$;

which was shown later to be superfluous.

We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $X = (X, \Gamma) = (X, X; \Gamma)$. For an $(X, D; \Gamma)$, a subset C of X is said to be *G-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

For a nonempty subset B of X, the *G-convex hull* of B, $G\text{-co}\, B$, is defined by

$$G\text{-co}\, B = \bigcap \{Y : B \subset Y \subset X \text{ and } Y \text{ is } G\text{-convex}\}.$$

For a G-convex space $(X, D; \Gamma)$, a map $T : D \multimap X$ is called a *G-KKM map* if $\Gamma_N \subset T(N)$ for each $N \in \langle D \rangle$.

Let I be a nonempty index set and $(X, D; \Gamma)$ a G-convex space. A map $T : I \multimap X$ is called a *generalized KKM map* if for each $J \in \langle I \rangle$, there eixsts a function $\sigma : J \to D$ such that for any $M \in \langle J \rangle$, we have $\Gamma_{\sigma(M)} \subset T(M)$; see [P3], [PK5].

The following is basic in this paper:

Theorem 0. [PK2,3, Theorem 1] *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ maps, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *for each $x \in D$, Sx is compactly open in Y;*

(2) *for each $y \in F(X)$, $M \in \langle S^- y\rangle$ implies $\Gamma_M \subset T^- y$;*

(3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*

(4) *either*

  (i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D\rangle$; or*

  (ii) *for each $N \in \langle D\rangle$, there exists a compact G-convex subset L_N of X containing N such that $F(L_N)\backslash K \subset S(L_N \cap D)$.*

Then there exists an $\overline{x} \in X$ such that $F\overline{x} \cap T\overline{x} \neq \emptyset$.

The admissible class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ are now well-known; see [P4, PK1,2]. A subclass of $\mathfrak{A}_c^\kappa$ is that of acyclic maps $\mathbb{V}$; that is, $T \in \mathbb{V}(X, Y)$ is an u.s.c. map with nonempty compact acyclic values.

Note that if $F = f \in \mathbb{C}(X, Y)$, a single-valued continuous map, then Hausdorffness of Y in Theorem 0 is not necessary.

The following Fan-Browder fixed point theorem is inmediately obtained from Theorem 0.

Theorem 1. *Let X be a compact G-convex space and $S : X \multimap X$ a map satisfying*

(1) *for each $y \in X$, $S^- y$ is G-convex; and*

(2) *$\{\operatorname{Int} Sx : x \in X\}$ covers X.*

Then S has a fixed point $x_0 \in X$; that is, $x_0 \in Sx_0$.

Here, we should mention that, in [PK4], there are lots of typos, especially, in the references.

1. On generalizations of the Ky Fan section theorem

In this section, we show that main results of Yuan [Y] and Wu and Xu [WX] are consequences of Theorems 0 and 1.

X.-Z. Yuan [Y] gave some generalizations of Fan's section theorem or geometric lemma. Then as applications, some new Ky Fan minimax inequalities and fixed point theorems were added in [Y].

The following is Yuan's main theorem, which follows from Theorem 0:

Theorem 2. [Y, Theorem 2] *Let X be a Hausdorff space and Y be a nonempty convex subset of a Hausdorff topological vector space F. Suppose that A is a subset of $X \times Y$ and there exist a subset B of A and a nonempty compact subset K of X such that B is closed in $X \times Y$ and*

(a) *for each $y \in Y$, the set $\{x \in X : (x, y) \in A\}$ is closed in X;*
(b) *for each $x \in X$, the set $\{y \in Y : (x, y) \notin A\}$ is convex or empty;*
(c) *for each $y \in Y$, the set $\{x \in K : (x, y) \in B\}$ is nonempty and contractible.*

Then there exists $x_0 \in K$ such that $\{x_0\} \times Y \subset A$.

Proof. We apply Theorem 0 interchanging the roles of X and $Y = D$. Define $S = T : Y \multimap X$ by

$$Sy = Ty = \{x \in X : (x, y) \notin A\} \quad \text{for} \quad y \in Y,$$

and $F : Y \multimap K$ by

$$Fy = \{x \in K : (x, y) \in B\} \quad \text{for} \quad y \in Y.$$

Note that F is compact. Since $B \subset X \times Y$ is closed, $\mathrm{Gr}(F) \subset B^- \cap (Y \times K)$ is also closed in $Y \times K$. Therefore, F is u.s.c. and has nonempty closed contractible values by (c). Therefore, F is an acyclic map and hence $F \in \mathbb{V}(Y, X) \subset \mathfrak{A}_c^\kappa(Y, X)$.

We show that conditions (1), (2), and (4) of Theorem 0 are satisfied. In fact,

(1) for each $y \in Y$, Ty is open by (a);
(2) for each $x \in X$, $S^- x = T^- x$ is convex or empty by (b); and
(4) $F(Y) \subset K$ implies condition (4)(ii) trivially.

Suppose that condition (3) holds; that is, $K \subset S(Y) = T(Y)$. Then, by Theorem 0, there exists a $\overline{y} \in Y$ such that $F\overline{y} \cap T\overline{y} \neq \emptyset$; that is, $\overline{x} \in F\overline{y} \cap T\overline{y}$ for some $\overline{x} \in K$. Hence, $(\overline{x}, \overline{y}) \in B$ and $(\overline{x}, \overline{y}) \notin A$, which contradicts $B \subset A$.

Therefore (3) does not hold; that is, $K \not\subset T(Y)$, and there exists an $x_0 \in K$ such that $x_0 \notin Ty$ for all $y \in Y$, or equivalently, $(x_0, y) \in A$ for all $y \in Y$. This completes our proof.

Remarks. 1. A sharpened form of Theorem 2 was already obtained by Park [P2, Corollary 2.1], where Y is a mere convex space and acyclicity was adopted instead of contractibility in (c).

2. Far-reaching generalized (or similar) forms of Theorem 1 and other results in [Y] were already given by Park, Bae, and Kang [PBK].

3. A generalization of Theorem 2 for G-convex spaces was given in [PK3, Theorem 9], which is equivalent to Theorem 0.

In [WX], Wu and Xu gave a section theorem of Ky Fan type for H-spaces and, as its applications, obtained a minimax theorem, a coincidence theroem, and an intersection theorem for sets with H-convex sections. In fact, their work was mainly based on the H-space version of Theorem 1. Therefore, all of the results in [WX] can be extended to G-convex spaces by exploiting Theorem 1 and following the methods developed in [PK1-5].

2. On generalized S-KKM maps

The following definition is due to Lin and Chang [LC]:

Let X be a nonempty set, (Y, Γ) a G-convex space, and $S, T : X \multimap Y$. Then T is a *generalized S-KKM* map if for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, $G\text{-co}(\bigcup_{i=1}^n Sx_i) \subset \bigcup_{i=1}^n Tx_i$.

For a convex subset Y of a vector space, this defintion reduces to that of Chang and Yen [CY].

Moreover, Cheng [C] considered the following particular situation:

Let E and G be two topological vector spaces, $X \subset E$ an arbitrary set, $\varphi : X \to G$ a function, and $F : X \multimap G$ a map satisfying

$$\mathrm{co}\{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_k)\} \subset \bigcup_{i=1}^{k} Fx_i$$

for any finite subset $\{x_1, x_2, \ldots, x_k\} \subset X$.

We show that all those concepts can be considered in the frame of the G-convex space theory as follows:

Proposition 3. *A generalized S-KKM map can be regarded as a G-KKM map.*

Proof. Let X be a nonempty set, (Y, Γ) a G-convex space, and $S, T : X \multimap Y$. For each $x \in X$, choose a point $\overline{x} \in Y$ such that $\overline{x} \in Sx$. Let $D := \{\overline{x} \in Y : x \in X\}$. Then $S, T : X \multimap Y$ can be regarded $\overline{S}, \overline{T} : D \multimap Y$ as follows:

$$\overline{S}\overline{x} = Sx \quad \text{and} \quad \overline{T}\overline{x} = Tx \quad \text{for all} \quad x \in X.$$

Since (Y, Γ) is a G-convex space and T is a generalized S-KKM map, by defining

$$\Gamma'_A := G\text{-co}\,\overline{S}(A) \subset \overline{T}(A)$$

for each $A \in \langle D \rangle$, the triple $(Y, D; \Gamma')$ becomes a G-convex space and $\overline{T}$ is a G-KKM map. Note that $A \subset \overline{S}(A)$ and $G\text{-co}\,A \subset G\text{-co}\,\overline{S}(A) = \Gamma'_A$. If $|A| = n + 1$, there is a continuous map $\phi_A : \Delta_n \to \Gamma_A \subset G\text{-co}\,A \subset \Gamma'_A$ satisfying the definition of a G-convex space. This completes our proof.

Proposition 4. *A generalized S-KKM map is a generalized KKM map.*

Proof. Let X be a nonempty set, (Y, Γ) a G-convex space, and $S, T : X \multimap Y$. Suppose that T is a generalized S-KKM map. For each $J \in \langle X \rangle$, choose a function $\sigma : J \to Y$ such that $\sigma(x) \in Sx$ for $x \in J$. Then for any $M = \{x_1, \ldots, x_n\} \subset J$, we have

$$\Gamma_{\sigma(M)} = \Gamma(\{\sigma(x_1), \ldots, \sigma(x_n)\}) \subset G\text{-co}(\bigcup_{i=1}^{n} Sx_i) \subset T(M).$$

Therefore, T is a generalized KKM map.

Remark. For the study on generalized KKM maps, see [P1,3] and a forthcoming work [PK5].

3. Cheng's KKM theorem

In [C], C. Cheng generalized the minimax inequality of Brezis *et al.* [BNS] to the one involving two functions defined on the product of two topological vector spaces, and used it to discuss the existence of solutions of some variational inequalities. In order to prove his inequality, first Cheng gave a slight generalization of the KKM Lemma due to Fan.

In this section, we show that Cheng's KKM theorem is a simple consequence of an earlier result of the author and W. K. Kim [PW]. We add other comments related to those works.

Theorem 5. [PW, Theorem 1] *Let Y be a convex subset of a Hausdorff topological vector space E, and $\emptyset \neq X \subset Y$. Let $T : X \multimap E$ be a KKM-map such that each Tx is finite relatively closed subset of Y. Furthermore, assume the following:*

(1) *There exists a nonempty finite dimensional set $X_0 \subset X$, contained in some compact convex subset of Y, such that $\overline{\bigcap_{x \in X_0} Tx}$ is a compact subset of Y.*

(2) *For every line segment L of E we have*

$$\overline{\bigcap_{x \in X \cap L} Tx} \cap L = \bigcap_{x \in X \cap L} Tx \cap L.$$

Then $\bigcap_{x \in X} Tx \neq \emptyset$.

Remark. Theorem 5 generalizes the basic lemma of [BNS]. Note that those results are stated in old-fashioned way. In fact, we can regard Y as a convex space and Tx is closed in Y. Then (2) seems to be superfluous. Moreover, we can use more general coercivity condition than (1). A far reaching generalization of Theorem 5 is [PK2, Theorem 3].

In the following, we show that Cheng's main result is a consequence of Theorem 5:

Theorem 6. [C, Theorem 1] *Let E and G be two real Hausdorff topological vector spaces, $X \subset E$ be an arbitrary set, $\varphi : X \to G$ be a mapping and $F : X \multimap G$ be a map such that*

(1) *$\overline{Fx_0}$ is a compact subset of G for some $x_0 \in X$,*

(2) $\mathrm{co}\{\varphi(x_1), \ldots, \varphi(x_k)\} \subset \bigcup_{i=1}^k Fx_i$ *for any finite subset $\{x_1, \ldots, x_k\} \subset X$,*

(3) *the intersection of Fx with any finite-dimensional subspace of G is closed for any $x \in X$, and*

(4) *for any convex subset C of G,*

$$(\overline{\bigcap_{x \in \varphi^{-1}(C)} Fx}) \cap C = (\bigcap_{x \in \varphi^{-1}(C)} Fx) \cap C.$$

Then $\bigcap_{x\in X} Fx \neq \emptyset$.

Proof. Let $Y = E = G$ in Theorem 5 and $X' := \varphi(X) \subset Y$. Define a map $T : X' \multimap E$ by $T\varphi(x) = Fx$ for each $\varphi(x) \in X'$, where $x \in X$. Then T is a KKM-map by (2) and has finitely closed values. Putting $X_0 := \{x_0\}$, condition (1) of Theorem 5 is clearly satisfied. We show that condition (2) of Theorem 5 holds. Let L be a line segment of G. Then

$$\varphi(x) \in X' \cap L \Longleftrightarrow \varphi(x) \in \varphi(X) \cap L \Longleftrightarrow x \in \varphi^{-1}(L)$$

and hence

$$\bigcap_{\varphi(x)\in X'\cap L} T\varphi(x) = \bigcap_{x\in\varphi^{-1}(L)} Fx$$

by (4). Therefore, all of the requirements of Theorem 5 with X' replacing by X are satisfied. Therefore, $\bigcap_{\varphi(x)\in X'} T\varphi(x) \neq \emptyset$ holds, which readily implies $\bigcap_{x\in X} Fx \neq \emptyset$.

Remarks. Similarly, other results in [C] are all simple consequences of known theorems.

4. A Sehgal-Singh-Watson type theorem

In 1981, Komiya [K] defined a convexity on topological spaces without linear structures paying attention to the concept of convex hull. Then some theorems which had been obtained in topological vector spaces were given in the frame of this convexity. In fact, the Fan-Browder fixed point theorem, Fan's theorem on systems of convex inequalities, and Sion's minimax theorem were extended to this convexity.

Hadžić [H] followed Komiya's study and obtained generalizations of Tarafdar's extension [Ta] of the Fan-Browder theorem and a coincidence theorem due to

Sehgal, Singh, and Watson [SSW1, 2] in the frame of Komiya's convexity. In this frame, Arandjelović [A] obtained generalizations of the KKM theorem and the Fan minimax theorem.

Later in 1993, we noted that Komiya's convexity is an example of generalized convex spaces; see [PK1-4]. Further, in [PK3], we established the foundations of the KKM theory for these new spaces, which were shown to include all of the above results on Komiya's convexity except the Sehgal-Singh-Watson type theorem in [H].

In [SSW1, 2], the authors obtained a coincidence theorem and its direct consequences. In [P5], we obtained an improved version of their result. Now in this section, we obtain a G-convex space version of the coincidence theorem as follows:

Theorem 7. *Let (X, Γ) be a G-convex space, such that Γ_A is compact for each $A \in \langle X \rangle$, Y a topological space, and $T, S : X \multimap Y$ maps satisfying the following:*

(1) *T is a compact l.s.c. map with nonempty values; and*

(2) *S has open values, $S^- y$ is nonempty for each $y \in \overline{T(X)}$, and $(S^- T)x$ is G-convex for each $x \in X$.*

Then T and S have a coincidence point.

Proof. For each $y \in \overline{T(X)}$, there exists an $x \in S^- y$ by (2). Since $y \in Sx$, we have $\overline{T(X)} \subset S(X)$. Since each Sx is open and $\overline{T(X)}$ is compact, we have $T(X) \subset \bigcup_{i=1}^n Sx_i$ for some $\{x_1, \ldots, x_n\} \in \langle X \rangle$. Therefore, we have $X = \bigcup_{i=1}^n (T^- S)x_i$.

Let $X' := \Gamma(\{x_1, \ldots, x_n\})$ ad define a map $G : X' \multimap X'$ by $Gx = (T^- S)x \cap X'$ for $x \in X'$. We show that G satisfies the requirements of Theorem 1 with $X = X'$ and $S = G$.

(i) For each $z \in X'$,

$$\begin{aligned} G^- z &= \{x \in X' : z \in Gx\} = \{x \in X' : z \in (T^- S)x \cap X'\} \\ &= \{x \in X' : Tz \cap Sx \neq \emptyset\} = \{x \in X' : x \in (S^- T)z\} \\ &= (S^- T)z \cap X'. \end{aligned}$$

Since $Tz \neq \emptyset$, $(S^- T)z$ is nonempty and G-convex by (2). Therefore, $G^- z$ is G-convex in X', and condition (1) of Theorem 1 is satisfied.

(ii) Since $X = \bigcup_{i=1}^n (T^- S)x_i$, we have

$$X' = X \cap X' = \bigcup_{i=1}^n [(T^- S)x_i \cap X'] = \bigcup_{i=1}^n Gx_i.$$

We show that each Gx_i is open in X'. In fact, for each i, $Gx_i = (T^- S)x_i \cap X'$. Since Sx_i is open and T is l.s.c., $(T^- S)x_i$ is open in X and hence Gx_i is open in X'. Therefore, condition (2) of Theorem 1 holds.

Since X' is a compact G-convex space, by Theorem 1, G has a fixed point $x_0 \in X'$; that is, $x_0 \in (T^- S)x_0$, or $Tx_0 \cap Sx_0 \neq \emptyset$. This completes our proof.

Note that for a convex space X, Theorem 7 reduces to Park [P5, Theorem 2], and for a Komiya type convex space X, to Hadžić [H, Theorem 4].

5. On a paper of S. Zhang and X. Zhang

In [ZZ], the authors claimed to prove a new class of KKM theorem. As applications, they used their result to obtain some matching theorem, coincidence theorem, and section theorem. The results presented in [ZZ] were claimed to be a unifications and generalization of the corresponding results in ten papers including [P1].

They began with the following:

Lemma. [ZZ, Lemma 1.1] *Let E be a linear space with finite (dimensional Euclidean) topology, Y a topological space, $A_1, \ldots, A_n$ be n compactly closed subsets of Y and $Y = \bigcup_{i=1}^n A_i$. Then for any n points $u_1, \ldots, u_n \in E$ and for any $S \in \mathbb{C}(\mathrm{co}\{u_1, \ldots, u_n\}, Y)$ (the set of all continuous functions from $\mathrm{co}\{u_1, \ldots, u_n\}$ to Y), there exist m, $1 \leq m \leq n$, and $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ such that*

$$S(\mathrm{co}\{u_{i_1}, \ldots, u_{i_m}\}) \cap \bigcap_{j=1}^m A_{i_j} \neq \emptyset.$$

Let us compare this lemma with the following:

Theorem 8. [P1, Theorem 7] *Let X be a convex space, Y a topological space, and $s \in \mathbb{C}(X,Y)$. Let $A_1, A_2, \dots, A_n$ be a finite family of n compactly closed subsets of Y such that $\bigcup_{i=1}^n A_i = Y$. Then for any n points $x_1, x_2, \dots, x_n$ (not necessarily distinct) of X, there exist k indices (for a suitable k) $i_1 < i_2 < \cdots < i_k$ such that*

$$s(\mathrm{co}\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}) \cap \bigcap_{j=1}^{k} A_{i_j} \neq \emptyset.$$

Proof of Lemma using Theorem 8. Put $X = \mathrm{co}\{u_1, \dots, u_n\}$. Then X is a convex space.

Similarly, [ZZ, Theorem 2.1], which is a main theorem of [ZZ], is a simple consequence of [P1, Theorems 3,4, and 8].

Further, the authors of [ZZ] claimed that [ZZ, Theorem 2.1] unifies and extends many kinds of KKM theorems in other's works. They also claimed that their method is more simple and more intuitive. Note that their claims are entirely incorrect.

Finally, in [CW], S.-S. Chang (= S. Zhang) and X. Wu claimed that each of their Theorems 2.2 and 2.3 is a (improvement and) generalization of Theorem 1 of Park [P1]. This claim is also entirely incorrect.

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