FIXED POINTS AND OPENNESS OF MULTIFUNCTIONS

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Abstract. Nguyen's results [N] for the openness of a map $A : X \to E$, where X is a convex subset of a locally convex Hausdorff topological vector space E, are extended to a very large class of multifunctions which appear in nonlinear analysis and algebraic topology.

1. Introduction

In [R3] Reichbach considered the problem: Let $A : X \to Y$ be a map of a topological space X into a topological space Y. Under what conditions is $A(X)$ open in Y ?

This is closely related to the surjectivity of a map. In fact, if $A(X)$ is closed and open in Y and if Y is connected, then we obtain $A(X) = Y$. Moreover, the problem of openness of a map in a Banach space or a locally convex Hausdorff topological vector space has been considered by a number of authors. See [CN, HNR, KN, N, R1-3]. Especially, Reichbach [R3] gave a particular solution of this problem in the case of selfmaps $A : E \to E$ of a Banach space E and some examples.

Later Nguyen [N] extended and improved Reichbach's result to locally convex Hausdorff topological vector spaces. Note that the methods in [R3, N] depend on

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fixed point theorems, which are generalizations of the well-known Schauder and Tychonoff type results. Actually those theorems are concerned with compact maps $A : X \to E$, where X is a convex subset of a topological vector space E. In this case, the whereabouts of the functional values Ax for x in the boundary of X play an important role.

Motivated by recent developments of fixed point theory for multifunctions, in this paper, we generalize Nguyen's results in several ways. First of all, we consider multifunctions in a very large class containing most of well-known multifunctions which appear in nonlinear analysis and algebraic topology. Moreover, we adopt the so-called Leray-Schauder condition as the boundary condition for those multifunctions.

2. Preliminaries

A *multifunction* or *map* $F : X \to 2^Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y; that is, a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. A multifunction $F : X \to 2^Y$ is *compact* provided $F(X)$ is contained in a compact subset of Y. Given two maps $F : X \to 2^Y$ and $G : Y \to 2^Z$, the *composite* $GF : X \to 2^Z$ is defined by $(GF)x = G(Fx)$ for $x \in X$.

For topological spaces X and Y, a map $F : X \to 2^Y$, is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^{-}(B)$ is closed in X. Note that composites of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact. Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

In a topological vector space any convex hulls of its finite subsets will be called *polytopes.*

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ values) [Gr], the acyclic maps $\mathbb{V}$ (with acyclic values), the O'Neill maps $\mathbb{N}$ (with values consisting of one or m acyclic components, where m is fixed) [Gr], the approachable maps $\mathbb{A}$ in a t.v.s. [BD], admissible maps in the sense of Górniewicz [G], permissible maps of Dzedzej [D], and others. For details, see [PK].

We introduce two more classes:

$F \in \mathfrak{A}_c^\sigma(X,Y) \Longleftrightarrow$ for any σ-compact subset K of X, there is an $\tilde{F} \in \mathfrak{A}_c(K,Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X,Y) \Longleftrightarrow$ for any compact subset K of X, there is an $\tilde{F} \in \mathfrak{A}_c(K,Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

Any class $\mathfrak{A}_c^\kappa$ will be called *admissible*. For details, see [P2, PK].

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Examples of $\mathfrak{A}_c^\sigma$ are $\mathbb{K}_c^\sigma$ due to Lassonde [L] and $\mathbb{V}_c^\sigma$ due to Park, Singh, and Watson [PSW]. Note that $\mathbb{K}_c^\sigma$ contains classes $\mathbb{K}$, Fan-Browder type maps, and $\mathbb{T}$. See [L].

Throughout this paper, E will denote a locally convex Hausdorff topological vector space and $S(E)$ the family of all continuous seminorms on E defining the topology of E. Let Bd, Int, and $\overline{\quad}$ denote the boundary, interior, and closure, resp.

For a $p \in S(E)$, $x_0 \in E$, and $r > 0$, we define

$$B_p(x_0,r) := \{x \in E : p(x - x_0) \le r\},$$
$$S_p(x_0,r) := \{x \in E : p(x - x_0) = r\}.$$

We say that a map $F : E \to 2^E$ is *p-compact* if for a certain closed convex balanced neighborhood U of the origin $0 \in E$ and the gauge p of U (that is, $U = \{x \in E : p(x) \le 1\}$, the sets $\overline{F(nU)}$ is compact for $n = 1, 2, \ldots$, where $nU = \{x \in E : p(x) \le n\}$. Note that for a single-valued (continuous) map $f : E \to E$, p-compactness is usually called completely continuous.

The following fixed point theorem is recently due to the author [P1-3].

Theorem 1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E, and $F \in \mathfrak{A}_c^\sigma(X, X)$. If F is compact, then there exists an $x_0 \in X$ such that $x_0 \in Fx_0$.*

Note that Theorem 1 contains a large number of historically well-known fixed point theorems.

3. Fixed Point of Multifunctions Satisfying the Leray-Schauder Condition

From Theorem 1, we obtain the following theorem for compact admissible maps satisfying the Leray-Schauder condition (LS).

Theorem 2. *Let E be a locally convex Hausdorff topological vector space, and U a convex neighborhood of the origin 0 of E. Then any compact map $F \in \mathfrak{A}_c^\sigma(\overline{U}, E)$ satisfying*

(LS) $Fx \cap \{\lambda x : \lambda > 1\} = \emptyset$ *for each* $x \in \operatorname{Bd} U$

has a fixed point.

Proof. Let p be the Minkowski functional of U. Since $0 \in \operatorname{Int} U$, p is continuous. Define $r : E \to \overline{U}$ by $r(x) = x$ for $x \in \overline{U}$ and $r(x) = p(x)^{-1}x$ for $x \notin \overline{U}$; that is,

$$r(x) = [\max\{1, p(x)\}]^{-1}x \quad \text{for } x \in E.$$

Then r is a continuous retraction of E onto $\overline{U}$. Define $G = r \circ F$. Then $G \in \mathfrak{A}_c^\sigma(\overline{U}, \overline{U})$ and is compact. Therefore, by Theorem 1, there exists an $x \in \overline{U}$ such that $x \in Gx$; that is, $x = r(y)$ for some $y \in Fx$. There are two possibilities: (i) $x \in \operatorname{Int} U$ or (ii) $x \in \operatorname{Bd} U$.

Suppose (i) holds. Then

$$1 > p(x) = p(r(y)) = [\max\{1, p(y)\}]^{-1}p(y).$$

Hence $p(y) < 1$ and this implies $r(y) = y$. Then $x = r(y) = y \in Fx$.

Suppose (ii) holds. Then

$$1 = p(x) = p(r(y)) = [\max\{1, p(y)\}]^{-1} p(y).$$

If $p(y) < 1$, we have a contradiction. If $p(y) > 1$, then $x = r(y) = p(y)^{-1}y$ and $y = p(y)x \in Fx$. This contradicts (LS). Thus $p(y) = 1$, so that $r(y) = y$ and $x = r(y) = y \in Fx$. This completes our proof.

Note that Theorem 2 also contains a large number of known fixed point theorems. See [P3], where generalized forms of Theorem 2 are given. The so-called Leray-Schauder condition (LS) seems to be originated from Schaefer [S].

Theorem 3. *Let E be a locally convex Hausdorff topological vector space, $p \in S(E)$, $F \in \mathfrak{A}_c^\sigma(E, E)$ a p-compact map, $x_0 \in E$ and $r > 0$. If*

(LS)′ $(Fx - x_0) \cap \{\lambda(x - x_0) : \lambda > 1\} = \emptyset$ *for each* $x \in S_p(x_0, r)$,

then F has a fixed point in $B_p(x_0, r)$.

Note that the p-ball $B_p(x_0, r)$ is a convex neighborhood of x_0 and that the boundary of $B_p(x_0, r)$ is precisely the p-sphere $S_p(x_0, r)$. Therefore, Theorem 3 clearly follows from Theorem 2. Recently, particular forms of Theorem 3 for normed vector spaces were due to Granas [Gr, Theorems 2.2-2.3].

If $F(S_p(x_0, r)) \subset B_p(x_0, r)$, then (LS)′ clearly holds. In this case, for $F = f \in \mathbb{C}(E, E)$, Theorem 3 reduces to Nguyen [N, Theorem 1].

4. Openness of Multifunctions Related to Admissible Maps

From Theorem 3, we have the following :

Theorem 4. *Let E be a locally convex Hausdorff topological vector space, $p \in S(E)$ and $A : E \to 2^E$ a map such that, for some $\beta \neq 0$ and $\overline{y} \in E$,*

(i) *$G \in \mathfrak{A}_c(E, E)$ defined by $Gx = x - \beta Ax$ for $x \in E$ is p-compact; and*

(ii) *$F \in \mathfrak{A}_c(E, E)$ defined by $Fx = x - \beta(Ax - \overline{y})$ satisfies* (LS)$'$.

Then, for each $x_0 \in E$ and $r > 0$, there exists a point $\overline{x} \in B_p(x_0, r)$ such that $\overline{y} \in A\overline{x}$.

Proof. We apply Theorem 3 to F. In fact, since G is p-compact, it follows that F is also p-compact. Moreover, F satisfies (LS)$'$. Therefore, there exists an $\overline{x} \in B_p(x_0, r)$ such that $\overline{x} \in F\overline{x}$. Since $\beta \neq 0$, we have $0 \in \beta(A\overline{x} - \overline{y})$ and $\overline{y} \in A\overline{x}$.

For a map $A \in \mathbb{C}(E, E)$ satisfying $A(S_p(x_0, r)) \subset B_p(x_0, r)$ instead of (LS)$'$, Theorem 4 reduces to Nguyen [N, Corollary].

Let X and Y be topological spaces, $A : X \to 2^Y$ a map, and $y_0 \in A(X)$. We say that A is *open* at y_0 if there exists a neighborhood V of y_0 in Y such that $V \subset A(X)$.

Theorem 5. *Let E be a locally convex Hausdorff topological vector space, $p \in S(E)$, $A : E \to 2^E$, and $y_0 \in A(E)$. Suppose that there exists an $r_0 > 0$ such that*

(iii) *for each $\overline{y} \in B_p(y_0, r_0)$, there exist an $x_0 \in A^{-1}\overline{y}$, a $\beta \neq 0$, and an $r > 0$ satisfying* (i) *and* (ii).

Then $B_p(y_0, r_0) \subset A(E)$; that is, A is open at the point $y_0 \in A(E)$.

Proof. From Theorem 4, for every $\overline{y} \in B_p(y_0, r_0)$, there exists an $\overline{x} \in B_p(y_0, r_0)$ such that $\overline{y} \in A\overline{x}$. Therefore, every $\overline{y} \in B_p(y_0, r_0)$ belongs to the range $A(E)$ of A, and hence A is open at the point $y_0 \in A(X)$.

For a map $A \in \mathbb{C}(E, E)$, Theorem 5 reduces to [N, Theorem 2].

In order to give an illustration of Theorem 5, we give the following :

For $G \in \mathfrak{A}_c(E, E)$, let

$$N_p(x_0, r) = \sup\{p(u - u_0) : u \in Gx,\ u_0 \in Gx_0,\ x \in S_p(x_0, r)\}$$

for $p \in S(E)$, $x_0 \in E$, and $r > 0$. If G is p-compact, then $N_p(x_0, r)$ is bounded.

Theorem 6. *Let E be a locally convex Hausdorff topological vector space, $p \in S(E)$, $G \in \mathfrak{A}_c(E, E)$ a p-compact map, and $A : E \to 2^E$ given by $Ax = x - Gx$ for $x \in E$. If there exist an $x_0 \in E$ and an $r > 0$ such that*

$$N_p(x_0, r) < r, \tag{1}$$

then A is open at any $y_0 \in Ax_0$.

Proof. Since $G : x \mapsto x - Ax$ is p-compact, condition (i) is satisfied with $\beta = 1$. Let $\overline{y} \in E$ and $Fx = Gx + \overline{y} = x - [Ax - \overline{y}]$ for $x \in E$. Since G is p-compact, there exists a closed, convex, balanced neighborhood U of 0 in E satisfying $U = \{x \in E : p(x) \leq 1\}$ such that $\overline{G(nU)}$ is compact for each $n = 1, 2, 3, \ldots$. For any $x \in S_p(x_0, r)$, we have

$$\begin{aligned} \sup_{z \in Fx} p(z - x_0) &= \sup_{y \in Ax} p(x - y + \overline{y} - x_0) \\ &\leq \sup_{y \in Ax} p((x - y) - (x_0 - y_0)) + p(y_0 - \overline{y}), \end{aligned} \tag{2}$$

where $y_0 \in Ax_0$ is arbitrary, $x - y \in Gx$, and $x_0 - y_0 \in Gx_0$. Let r be as in (1) and $r_1 < r$ such that $N_p(x_0, r) \leq r_1 < r$. Choose r_0 such that $0 < r_0 < r - r_1$. Then by (2), for any $\overline{y} \in B_p(y_0, r_0)$, we have $\sup_{z \in Fx} p(z - x_0) \leq r_1 + (r - r_1) = r$. Therefore, $F(S_p(x_0, r)) \subset B_p(x_0, r)$ and hence (LS)$'$ holds. This shows condition (ii).

Therefore, by Theorem 5, $B_p(y_0, r_0) \subset A(E)$ and hence A is open at any $y_0 \in Ax_0$.

For a map $G = g \in \mathbb{C}(E, E)$, Theorem 6 reduces to Nguyen [N, Theorem 3], which extends and improves Reichbach [R3, Theorem 1]. Note that all the results in [R3] hold for normed vector spaces instead of Banach spaces. Moreover, some examples of Theorem 6 can be seen in [R3, N]. Further, if $A \in \mathbb{C}(E, E)$ is linear in Theorems 5 and 6, then $A(E)$ is an open linear subset of E and hence we have $A(E) = E$ as noted in [R3, N].

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