REMARKS ON SOME COINCIDENCE THEOREMS

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Abstract. It is shown that a new coincidence theorem due to Wu is a consequence of our earlier result. We obtain far-reaching generalizations and related results.

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In 1991, the author [P1] first showed that any compact acyclic multimap from a nonempty convex subset of a locally convex Hausdorff topological vector space into itself has a fixed point, where an acyclic multimap is an upper semicontinuous map with nonempty compact acyclic values. This result generalizes and unifies the historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, Himmelberg, and others. Note that all of those authors were concerned with single-valued maps or convex-valued multimaps. For the literature, see [P1,2,5].

Until now a number of authors incorrectly claimed to obtain our theorem or its generalizations.

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In this paper, every topological space is assumed to be Hausdorff and t.v.s. means topological vector spaces. For topological spaces X and Y, we denote

$f \in \mathbb{C}(X, Y) \Longleftrightarrow f : X \to Y$ is a single-valued continuous map.

$F \in \mathbb{V}(X, Y) \Longleftrightarrow F : X \multimap Y$ is an acyclic multimap; that is, an upper semicontinuous multimap with acyclic compact values.

$F \in \mathbb{V}_c(X, Y) \Longleftrightarrow F : X \multimap Y$ is a finite composition of acyclic maps where the intermediate spaces are topological.

For other notations and terminology, we follow [P4,5, PSW].

Recently, the following generalization of our 1991 theorem appeared.

Theorem 1. (Wu [W]) *Let X be a compact space and Y a nonempty convex subset of a locally convex t.v.s. Suppose that $f \in \mathbb{C}(X, Y)$ and $T \in \mathbb{V}(Y, X)$. Then there exists a point $x_0 \in X$ such that $x_0 \in T(fx_0)$.*

In fact, for $X \subset Y$ and $f = i : X \hookrightarrow Y$, Theorem 1 reduces to our theorem. Moreover, Theorem 1 is called a new coincidence theorem because its conclusion is equivalent to the following : there exist points $x_0 \in X$ and $y_0 \in Y$ such that $y_0 = fx_0$ and $x_0 \in Ty_0$.

However, Theorem 1 is a simple consequence of the following known result:

Theorem 2. (Park, Singh, and Watson [PSW]) *Let X be a nonempty convex subset of a locally convex t.v.s. and $F \in \mathbb{V}_c(X, X)$. If F is compact, then F has a fixed point.*

Proof of Theorem 1 using Theorem 2. Since $f \in \mathbb{C}(X, Y) \subset \mathbb{V}(X, Y)$ and $T \in \mathbb{V}(X, Y)$, it follows that $f \circ T \in \mathbb{V}_c(Y, Y)$. Since T is a compact multimap, so is $f \circ T$. Therefore, by Theorem 2, $f \circ T$ has a fixed point $y_0 \in Y$; that is $y_0 \in f(Ty_0)$. Then there exists an $x_0 \in Ty_0 \subset X$ such that $y_0 = fx_0$. This implies $x_0 \in Ty_0 = T(fx_0)$.

Note that, in Theorem 1, we can replace $T \in \mathbb{V}(Y, X)$ by $T \in \mathbb{V}_c(Y, X)$ and then, obtain a generalized, but equivalent, form of Theorem 2.

This fact can be extended to much general situation by adopting the better admissible maps and admissible subsets (in the sense of Klee); see [P4,5].

A *convex space* X (in the sense of Lassonde) is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on convex hulls of its nonempty finite subsets. Such convex hulls are called *polytopes.* Note that any polytope is admissible.

For any convex space X and any topological space Y, the better admissible class $\mathfrak{B}$ of multimaps is defined as follows:

$F \in \mathfrak{B}(X, Y) \Longleftrightarrow F : X \multimap Y$ is a multimap such that for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P) : P \multimap P$ has a fixed point.

The following is a known generalization of Theorem 2:

Theorem 3. (Park [P4,5]) *Let E be a t.v.s. and X an admissible convex subset of E. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

From Theorem 3, we obtain a far-reaching generalization of Theorem 1:

Theorem 4. *Let X be a compact space and Y an admissible (in the sense of Klee) convex subset of a t.v.s. Suppose that $f \in \mathbb{C}(X, Y)$ and $T \in \mathfrak{B}(Y, X)$ is a closed map. Then there exists a point $x_0 \in X$ such that $x_0 \in T(fx_0)$.*

Proof. Since $T \in \mathfrak{B}(Y, X)$ is closed and compact and $f \in \mathbb{C}(Y, X)$, $f \circ T$ is closed and compact. Therefore $f \circ T \in \mathfrak{B}(Y, Y)$; see [P3, Fact (c) and Proposition (v)]. Hence, by Theorem 3, $f \circ T$ has a fixed point $y_0 \in Y$; that is, $y_0 \in f(Ty_0)$. It follows that $y_0 = fx_0$ for some $x_0 \in Ty_0 \subset X$ and hence $x_0 \in T(fx_0)$. This completes our proof.

Note that, for $X \subset Y$ and $f = i : X \hookrightarrow Y$, Theorem 4 reduces to Theorem 3.

In Theorem 4, we may assume that $f \in \mathbb{C}(T(Y), Y)$ and $T \in \mathfrak{B}(Y, T(Y))$ without affecting its proof.

Theorem 4 can be used to obtain the following coincidence theorem:

Theorem 5. *Let X be a compact space, Y a convex space, and $T \in \mathfrak{B}(Y,X)$ a closed map. Suppose that $F, G : X \multimap Y$ are two multimaps such that*

(i) *for each $x \in T(Y)$, $\operatorname{co} Gx \subset Fx$; and*

(ii) $\overline{T(Y)} \subset \bigcup\{\operatorname{Int} G^{-}y : y \in Y\}$.

Then there exists a point $x_0 \in X$ such that $x_0 \in T(Fx_0)$.

Proof. Since $\overline{T(Y)} \subset X$ is compact and covered by $\{\operatorname{Int} G^{-}y : y \in Y\}$, there is a finite set $N := \{y_1, y_2, \ldots, y_n\}$ in Y such that $\overline{T(y)} \subset \bigcup\{\operatorname{Int} G^{-}y : y \in N\}$. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the partition of unity corresponding to this cover, and $P := \operatorname{co} N$. Define $f : \overline{T(Y)} \to P$ by

$$fx = \sum_{1}^{n} \lambda_i(x) y_i = \sum_{i \in N_x} \lambda_i(x) y_i$$

for $x \in \overline{T(Y)}$, where $i \in N_x$ if and only if $\lambda_i(x) \neq 0$, which implies that $x \in \operatorname{Int} G^{-}y_i$. Then, for $i \in N_x$, we have $y_i \in Gx$. Clearly f is continuous and by (i) we have $fx \in \operatorname{co}\{y_i : i \in N_x\} \subset \operatorname{co} Gx \subset Fx$ for each $x \in T(Y)$. Since $f \in \mathbb{C}(T(Y), P)$ and $T|_P \in \mathfrak{B}(P, T(Y))$, by Theorem 4, there exists a point $x_0 \in X$ such that $x_0 \in T(fx_0) \subset T(Fx_0)$. This completes our proof.

If $\mathfrak{B}$ is replaced by its subclass $\mathbb{V}_c$, then Theorem 5 reduces the main theorem of Park *et al.* [PSW, Theorem 1]. Similarly, other results in that paper can be improved.

For $X = Y$ and $T = 1_X$, the identity map, Theorem 5 reduces to the following Fan-Browder fixed point theorem:

Theorem 6. *Let X be a compact convex space and $F, G : X \multimap X$ multimaps satisfying*

(i) *for each $x \in X$, $\operatorname{co} Gx \subset Fx$; and*

(ii) $X = \bigcup\{\operatorname{Int} G^{-}y : y \in X\}$.

Then F has a fixed point.

For the history and literature on Theorem 6, see [P1,2].

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