FIXED POINTS OF MULTIMAPS ON ORDERED SPACES

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Abstract. Let $F : X \multimap X$ be a multimap on an arc X or, more generally, a connected ordered space X with two end points. If F has connected graph, then F has a fixed point. We deduce several consequences from our new fixed point theorem and a generalization of the Bolzano intermediate value theorem.

Key Words and Phrases. (Generalized) arc, multimap (map), u.s.c., l.s.c., closed map, compact map.

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1. Introduction and preliminaries

Let X be a connected ordered space with two end points. Our main result in this paper is that a multimap $F : X \multimap X$ having connected graph has a fixed point $\overline{x} \in X$; that is, $\overline{x} \in F(\overline{x})$. We give some consequences of our new theorem and a generalization of the Bolzano intermediate value theorem. Finally, examples and remarks are added.

For topological spaces X and Y, a *multimap* or *map* $F : X \multimap Y$ is a function from X into the power set of Y with nonempty *values* $F(x)$ for $x \in X$ and *fibers* $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for $y \in Y$.

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A map $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X; *lower semi continuous* (l.s.c.) if for each open set $B \subset Y$, the set $F^{-1}(B)$ is open in X; *closed* if its graph $\mathrm{Gr}(F) = \{(x, y) : x \in X,\ y \in F(x)\}$ is closed in $X \times Y$; and *compact* if the closure $\overline{F(X)}$ of its range $F(X) = \bigcup\{F(x) : x \in X\}$ is compact in Y.

A linearly ordered set $(X, \leq)$ is called an *ordered space* if it has the order topology whose subbase consists of all sets of the form $\{x \in X : x < s\}$ and $\{x \in X : x > s\}$ for $s \in X$. Note that an ordered space X is connected iff it is Dedekind complete (that is, every subset of X having an upper bound has a supremum) and whenever $x < y$ in X, then $x < z < y$ for some z in X. For details, see Willard [W].

2. Main results

The following is our main result:

Theorem 1. *Let X be a connected ordered space with two end points and F : $X \multimap X$ a multimap. If F has connected graph, then F has a fixed point $\overline{x} \in X$; that is, $\overline{x} \in F(\overline{x})$.*

Proof. Let a and b be the end points of X such that $a < b$. Let

$$A = \{(s, t) \in X \times X : s < t\} \quad \text{and} \quad B = \{(s, t) \in X \times X : s > t\}.$$

Suppose that F has no fixed point. Then we have $x \notin F(x)$ for all $x \in X$ and hence

$$\mathrm{Gr}(F) \subset A \cup B.$$

Choose $y_0 \in F(a)$ and $y_1 \in F(b)$. Then

$$(a, y_0) \in A \cap \mathrm{Gr}(F) \quad \text{and} \quad (b, y_1) \in B \cap \mathrm{Gr}(F).$$

Since A, B are open and disjoint, this contradicts the connectedness of $\mathrm{Gr}(F)$. This completes our proof.

The class of maps having connected graph is quite large:

Lemma. (Hiriart-Urruty [H, Theorem 3.2]) *Let X, Y be a topological spaces, $C \subset X$ a connected subset, and $\Gamma : X \multimap Y$ be a multimap with connected values on C. Either of the next assumptions ensure that the graph of $\Gamma|_C$ is connected:*

(a) Γ *is l.s.c.*

(b) Γ *is u.s.c. and compact-valued.*

A map $\Gamma : X \multimap Y$ is called a *connectivity map* if the graph over each connected subset of X is a connected set. This concept was introduced by Nash for single-valued case; see Girolo [G].

From Theorem 1 and Lemma we have the following:

Corollary 2. *Let X be a connected ordered space with two end points. Then a map $F : X \multimap X$ has a fixed point if it satisfies one of the following conditions:*

(0) *F is a connectivity map.*

(i) *F is l.s.c. with connected values.*

(ii) *F is u.s.c. with compact connected values.*

(iii) *F has connected values and open fibers.*

(iv) *F is a closed compact map with connected values.*

Proof. (0) Since F is a connectivity map and X is connected, F has connected graph. Therefore, F has a fixed point by Theorem 1.

(i), (ii) By Lemma, F is a connectivity map. Therefore, (i) $\Longrightarrow$ (0) and (ii) $\Longrightarrow$ (0).

(iii) Since $F^{-1}(y)$ is open for each $y \in X$, F is l.s.c. Indeed, for each open set $A \subset X$, we have

$$F^{-1}(A) = \{x \in X : F(x) \cap A \neq \emptyset\} = \bigcup_{y \in A} F^{-1}(y)$$

is open. Therefore, (iii) implies (i).

(iv) It is well-known that a closed compact map is u.s.c. with compact values. Therefore, (iv) implies (ii).

Now the Bolzano intermediate value theorem can be extended as follows:

Theorem 3. *Let Z be a topological space, X a connected ordered space with two end points, and $F : Z \multimap X$ a map having connected graph. Let $x \in X$ be such that* $\inf F(Z) < x < \sup F(Z)$*. Then $x \in F(Z)$.*

Proof. Since $F(Z)$ is the projection of $\mathrm{Gr}(F)$ to X, $F(Z)$ is connected in X.

3. Examples and remarks

(1) We give some examples of connected ordered space X with two end points.

(a) $[0,1]$-spaces; that is, connected spaces admitting a continuous bijection onto the unit interval; see [R1], [P3].

(b) A (generalized) arc; that is, a continuum which has exactly two non-cut points. For example, the extended long line L^* constructed from the ordinal space $[0,\Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is an arc obtained from $[0,\Omega]$ by placing a copy of the interval $(0,1)$ between each ordinal α and its successor $\alpha+1$ and we give L^* the order topology; see [SS].

(2) Ricceri [R1] first considered $[0,1]$-spaces and, based on a new alternative principle for multimaps involving $[0,1]$-spaces, obtained new mini-max theorems in full generality and transparence. For further consequences of the principle, see successive works [R2, R3, C, CB, P3]. For $[0,1]$-spaces, Theorem 1 and Corollary 2 in Section 2 reduce to [P3, Theorem], which was deduced from the results in [R1].

(3) Theorems and Corollary work for any bounded closed interval $X=[a,b]$ or for

$$X=\{(0,0)\}\cup\{(x,y): x\in(0,1] \quad \text{and} \quad y=\sin\frac{1}{x}\}\subset\mathbb{R}^2.$$

Even for these spaces, Theorems 1 and 3 seem to be new. In fact, comparisons of Theorem 1 for $[0,1]$-spaces with known results were discussed in [P3].

(4) The connectedness of the graph in Theorems is essential: for example, for $X=[0,1]$, let $F=f: X\to X$ be given by

$$f(x)=1 \quad \text{if} \quad x\in[0,1/2] \quad \text{and} \quad f(x)=0 \quad \text{if} \quad x\in(1/2,1].$$

The connectedness of X in Corollary 2 is essential; for the $[0,1]$-space X given by

$$X=\{-1\}\cup(0,1)\cup\{2\}\subset\mathbb{R},$$

we give counterexamples of $F = f : X \to X$ violating cases (i)-(iv) as follows:

(i),(ii),(iv) $f(-1) = 2,\ f(2) = -1$ and $f(x) = \sqrt{x}$ for $x \in (0, 1)$.

(iii) $f(-1) = 2$, $f(2) = -1$, and $f(x) = -1$ for $x \in (0, 1)$.

(5) Corollary 2 (0) tells us that every connectivity map defined on a connected space X has connected graph, but not conversely. For example, consider the map $F : [-1, 1] \multimap [-1, 1]$ given by

$$F(x) = \{y \in [-1, 1] : y^2 = (x + 1)/2\} \quad \text{for} \quad x \in [-1, 1].$$

There are some fixed point theorems for connectivity maps; see [G]. It would be interesting to know whether these theorems can be extended to maps having connected graphs.

(6) Recently, the author [P1,2,4] has studied the fixed point theory of the better admissible class $\mathfrak{B}$ of multimaps in topological vector spaces. Now we give an example of a subclass of $\mathfrak{B}$ as follows:

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is the convex hull of a nonempty finite subset of X. We define the *"better" admissible class* $\mathfrak{B}$ of multimaps defined on X as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that for any polytope P in X and any continuous map $f : F(P) \to P$, the composition $f \circ (F|_P) : P \multimap P$ has a fixed point.

Let $X = Y = [a, b] \subset \mathbf{R}$ and $F : X \multimap X$ a multimap. *If F is a connectivity map, then $F \in \mathfrak{B}(X, X)$.* In fact, any polytope P of X is a subinterval of X. For any continuous map $f : F(P) \to P$, $f \circ (F|_P) : P \multimap P$ has connected graph and hence, has a fixed point by Theorem 1.

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