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FIXED POINT THEOREMS FOR NEW CLASSES OF MULTIMAPS

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Abstract. A closed multimap $T : X \multimap X$ having the KKM property has a fixed point whenever X is a compact convex set satisfying a certain topological condition. Our new result properly generalizes a known one due to W. K. Kim [4].

1. Introduction

Recently, in a sequence of papers, the author introduced the admissible classes $\mathfrak{A}_c^\kappa$ of multimaps, which are broad enough to include most of multimaps appearing in nonlinear analysis and algebraic topology. For the admissible classes, we established the foundations of the KKM theory [6,7] and the fixed point theory in topological vector spaces [5]. Some of those new results were, consecutively, applied to various topics. For the literature, see [9].

Apparently motivated by those works, Chang and Yen [2] extended the class $\mathfrak{A}_c^\kappa$ to multimaps having the KKM property and obtained some generalized results in the KKM theory and fixed point theory. On the other hand, W. K. Kim [4] gave a new fixed point theorem for lower semicontinuous multimaps in a Hausdorff topological vector space under an additional topological restriction. However, this result can be immediately improved by adopting the method in [2].

Recently, the author [8] showed that any closed compact multimap $T : X \multimap X$ having the KKM property has a fixed point whenever X is an admissible (in the sense of Klee) convex subset of a Hausdorff topological vector space. This result contains a large number of consequences including historically well-known results of Kakutani, Himmelberg, and many others.

In the present paper, we give another fixed point theorem on such maps T whenever X is a compact convex subset satisfying a certain topological condition. Our new theorem properly generalizes a result due to W. K. Kim [4].

We add some remarks related to the Schauder conjecture.

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2. Preliminaries

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y having nonempty values. The map $T : X \multimap Y$ induces two relations T^- and T^+ in $2^Y \times 2^X$ as follows: for $B \subset Y$,

$$T^-(B) := \{x \in X : Tx \cap B \neq \emptyset\} \quad \text{and} \quad T^+(B) := \{x \in X : Tx \subset B\}.$$

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *upper semicontinuous* if T^- sends closed sets to closed sets; *lower semicontinuous* if T^- sends open sets to open sets or, equivalently, if T^+ sends closed sets to closed sets; *continuous* if it is upper and lower semicontinuous; *closed* if its graph is closed in $X \times Y$; and *compact* if its range $T(X)$ is contained in a compact subset of Y.

Let cl and co denote closure and convex hull, respectively.

A *convex* space X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its nonempty finite subsets. Such convex hulls will be called *polytopes*.

The set of all nonempty finite subsets of a set X will be denoted by $\langle X \rangle$.

The following form of the KKM theorem is well-known:

LEMMA 1. *Let D be a nonempty subset of a convex space X and $G : D \multimap X$ a map with closed values such that* $\operatorname{co} A \subset G(A)$ *for each $A \in \langle D \rangle$. Then $\{Gx\}_{x \in D}$ has the finite intersection property.*

Let X be a convex space and Y a topological space. A map $T : X \multimap Y$ is said to *have the KKM property* provided that, for any map $S : X \multimap Y$ with closed values, if $\operatorname{co} A \subset T^+S(A)$ for each $A \in \langle X \rangle$, then $\{Sx\}_{x \in X}$ has the finite intersection property. See Chang and Yen [2]. Note that $Tx \subset Sx$ for each $x \in X$. Let $\mathfrak{K}(X, Y)$ denote the class of maps $T : X \multimap Y$ having the KKM property.

It was noted that, whenever Y is Hausdorff, $\mathfrak{K}(X, Y)$ contains the admissible class $\mathfrak{A}_c^\kappa(X, Y)$ due to the author. See [7, Corollary 2].

Throughout this paper, a t.v.s. means a Hausdorff topological vector space.

The following is due to Chang and Yen [2, Lemma 1], whose elegant proof is modified here for convenience of the readers:

LEMMA 2. *Let X be a convex subset of a t.v.s. E and V a symmetric open neighborhood of* 0 *in E. If $T \in \mathfrak{K}(X, X)$ is compact, then there exists an $x_V \in X$ such that $x_V \in Tx_V + \operatorname{co} V$.*

PROOF. Suppose that $x \notin Tx + \operatorname{co} V$ or, equivalently, $(x + \operatorname{co} V) \cap Tx = \emptyset$ for each $x \in X$. Let $Y = \operatorname{cl} T(X)$ be a compact subset of X. Define a map $S : X \multimap X$ by

$$Sx = Y \backslash \left(x + \frac{1}{4}(\operatorname{co} V) \right) \quad \text{for each} \quad x \in X.$$

Then each Sx is a closed subset of Y. Since

$$(Tx)^c \supset (x + \operatorname{co} V) \cap Y \supset (Sx)^c \quad \text{for each} \quad x \subset X,$$

where c denotes the complementation with respect to Y, we have $\bigcap_{x\in A} (Tx)^c \supset \bigcap_{x\in A} (Sx)^c$ and hence $T(A) \subset S(A)$ for any $A = \{x_1, x_2, \ldots, x_n\} \in \langle X\rangle$.

We claim that $T(\operatorname{co} A) \subset S(A)$.

(i) If $(Sx_1)^c \cap (Sx_i)^c = \emptyset$ for some i, $2 \leqq i \leqq n$, then

$$T(\operatorname{co} A) \subset Y = Sx_1 \cup Sx_i = S(A).$$

(ii) If $(Sx_1)^c \cap (Sx_i)^c \neq \emptyset$ for each i, $2 \leqq i \leqq n$, then

$$\left(x_1 + \frac{1}{4}(\operatorname{co} V)\right) \cap \left(x_i + \frac{1}{4}(\operatorname{co} V)\right) \neq \emptyset$$

implies

$$x_1 \in x_i + \frac{1}{2}(\operatorname{co} V) \quad \text{for all} \quad x_i \in A$$

and hence

$$x_1 \in x + \frac{1}{2}(\operatorname{co} V) \quad \text{for all} \quad x \in \operatorname{co} A.$$

Therefore,

$$(Sx_1)^c \subset \left(x_1 + \frac{1}{2}(\operatorname{co} V)\right) \cap Y \subset \bigcap_{x\in \operatorname{co} A} (x + \operatorname{co} V) \cap Y \subset \bigcap_{x\in \operatorname{co} A} (Tx)^c$$

and hence $T(\operatorname{co} A) \subset Sx_1 \subset S(A)$.

Since $T \in \mathfrak{K}(X, X)$, by definition, $\{Sx\}_{x\in X}$ has the finite intersection property in the compact set Y. Hence $\bigcap_{x\in X} Sx \neq \emptyset$. Let $x_0 \in \bigcap_{x\in X} Sx$. Then $x_0 \in Sx_0$, which contradicts the definition of S. This completes our proof.

From Lemma 2, Chang and Yen [2, Theorem 2] deduced the following fixed point theorem:

THEOREM A. *Let X be a nonempty convex subset of a locally convex t.v.s. Then every closed compact map $T \in \mathfrak{K}(X, X)$ has a fixed point $x_0 \in X$; that is, $x_0 \in Tx_0$.*

Note that, for the class $\mathfrak{A}_c^\kappa$ instead of $\mathfrak{K}$, Theorem A reduces to the previous result of the author [5-7]. Recently, Theorem A was generalized by the present author [8] as follows:

THEOREM B. *Let X be an admissible convex subset of a t.v.s. Then any closed compact map $T \in \mathfrak{K}(X,X)$ has a fixed point.*

Here, a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. There are many other examples of admissible sets. See Hadžić [3], Weber [10], and references therein.

We need the following:

LEMMA 3 [1, Lemma 4.1]. *Let X be a topological space, Y a subset of a t.v.s. E, and V a nonempty open subset of E. If a map $T : X \multimap Y$ is l.s.c., the map $T_V : X \multimap Y$ defined by $T_V x = (Tx + V) \cap Y$ for $x \in X$ has the open graph.*

LEMMA 4 [2, Proposition 3(ii)]. *Let X be a convex space and Y, Z topological spaces. If $T \in \mathfrak{K}(X,Y)$ and if $f : Y \to Z$ is continuous, then $fT \in \mathfrak{K}(X,Z)$.*

3. Main results

From Lemma 2, we obtain the following new fixed point theorem:

THEOREM 1. *Let X be a nonempty compact convex subset of a t.v.s. E and $T \in \mathfrak{K}(X,X)$. Suppose that*

(∗) if $y \in X$ satisfies $y \notin Ty + U$ for some symmetric open neighborhood U of 0 in E, then $y \notin \mathrm{cl}\{x \in X : x \in Tx + \mathrm{co}\,V\}$ for some symmetric open neighborhood V of 0 in E.

Then T has a fixed point.

PROOF. Let $\mathcal{V}$ be a local base of symmetric open neighborhoods of 0 in E and $V \in \mathcal{V}$. Then, by Lemma 2, there is an $x_V \in X$ such that $x_V \in Tx_V + \mathrm{co}\,V$. For each $U \in \mathcal{V}$, let

$$F_U := \{x \in X : x \in Tx + \mathrm{co}\,U\}, \quad F'_U := \{x \in X : x \in Tx + U\}.$$

Then $F_U \neq \emptyset$ and $F'_U \subset F_U$ for each $U \in \mathcal{V}$. It is clear that $\{F_U : U \in \mathcal{V}\}$ has the finite intersection property. Since each $\mathrm{cl}\,F_U$ is a closed subset of the compact space X, we have $\bigcap_{U\in\mathcal{V}} \mathrm{cl}\,F_U \neq \emptyset$. We claim that

$$\bigcap_{U\in\mathcal{V}} F'_U = \bigcap_{U\in\mathcal{V}} \mathrm{cl}\,F_U.$$

In fact, it suffices to show that the left hand side includes the right hand one. Suppose $y \in \bigcap_{U\in\mathcal{V}} \operatorname{cl} F_U$ satisfies $y \notin \bigcap_{U\in\mathcal{V}} F'_U$; that is, $y \notin Ty + U$ for some $U \in \mathcal{V}$. Then by condition $(*)$, there exists a $V \in \mathcal{V}$ satisfying $y \notin \operatorname{cl} F_V$. This is a contradiction. Therefore, there exists an $\hat{x} \in X$ such that

$$\hat{x} \in \bigcap_{U\in\mathcal{V}} F'_U = \bigcap_{U\in\mathcal{V}} \operatorname{cl} F_U \neq \emptyset.$$

Since

$$\hat{x} \in \bigcap_{U\in\mathcal{V}} (T\hat{x} + U) = T\hat{x} + \bigcap_{U\in\mathcal{V}} U = T\hat{x},$$

we have the conclusion.

In order to give a consequence of Theorem 1, we need the following:

PROPOSITION. *Let X be a nonempty compact convex subset of a t.v.s. E and $T : X \multimap X$ a lower semicontinuous map with convex values. Then $T \in \mathfrak{K}(X,X)$.*

PROOF. Let V be an open neighborhood of the origin 0 in E. Then the map $G : X \multimap E$ defined by $Gx = Tx + \operatorname{co} V$ has open graph by Lemma 3. Note that G has nonempty convex values and open fibers. Since X is compact, it is well-known that there exists a continuous selection $g : X \to E$ of G. Since $g \in \mathfrak{A}^\kappa_c(X,E) \subset \mathfrak{K}(X,E)$, we have clearly $G \in \mathfrak{K}(X,E)$.

Let $h : E \times E \to E$ be given by $h(a,b) = a - b$ for $(a,b) \in E \times E$ and $F : X \multimap E \times E$ by $Fx = Gx \times (\operatorname{co} V)$ for all $x \in X$. Then

$$(hF)x = h\big(Gx \times (\operatorname{co} V)\big) = Gx - \operatorname{co} V = Tx \subset X$$

for $x \in X$. Note that F has a continuous selection $f : X \to E \times E$ given by $fx = (gx, 0)$ for $x \in X$. Since $f \in \mathfrak{K}(X, E \times E)$ as a continuous map, we have $F \in \mathfrak{K}(X, E \times E)$. Since $h : E \times E \to E$ is continuous, by Lemma 4, we have $T = hF \in \mathfrak{K}(X,X)$. This completes our proof.

Note that the Proposition shows that the class $\mathfrak{K}$ properly contains $\mathfrak{A}^\kappa_c$ and that Theorem 1 properly generalizes the following:

THEOREM 2 [4, Theorem 1]. *Let X be a nonempty compact convex subset of a t.v.s. E and $T : X \multimap X$ a lower semicontinuous map with convex values satisfying condition $(*)$. Then T has a fixed point.*

As Kim noted in [4], when E is locally convex and T is single-valued, condition $(*)$ holds automatically, and hence Theorem 1 also extends theorems of Brouwer, Schauder, and Tychonoff.

4. Remarks on the Schauder conjecture

Our results in this paper are closely related to the following problem known as the Schauder conjecture, which is almost seventy years old.

PROBLEM 1. Does every compact convex subset X of a (metrizable) t.v.s. have the fixed point property? That is, does any continuous map $f : X \to X$ have a point $x_0 \in X$ such that $x_0 = f(x_0)$?

There are many open problems related to the Schauder conjecture. See [3], [10] and references therein. Note that Theorems B and 1 are quite general partial solutions to (generalized forms of) Problem 1.

If the following long-standing problem had a negative solution, then Theorem B would be the affirmative solution of the Schauder conjecture.

PROBLEM 2. Is there any (compact) convex nonadmissible subset of a t.v.s.?

Similarly, from Theorem 1, we can raise the following:

PROBLEM 3. Are there any compact convex subset of a t.v.s. E and any continuous map $T : X \to X$ not satisfying the condition $(*)$ in Theorem 1?

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