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Remarks on a Social Equilibrium Existence Theorem of G. Debreu

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Abstract—An acyclic version of the social equilibrium existence theorem of Debreu [1] is obtained. This is applied to deduce acyclic versions of theorems on saddle points, minimax theorems, and the Nash equilibrium. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Multimap (map), Closed map, Upper semicontinuous (u.s.c.), Lower semicontinuous (l.s.c.), Berge's theorem, Polyhedron, Acyclic, Equilibrium point, Saddle point, Minimax theorem, Nash equilibrium.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we give an acyclic version of the social equilibrium existence theorem of Debreu [1]. Our proof is much simpler than the original one. Moreover, our main result is applied to acyclic versions of a saddle point theorem, a minimax theorem, and the Nash equilibrium theorem.

For topological spaces X and Y, a *multimap* or *map* $F : X \multimap Y$ is a function from X into the power set 2^Y of Y with nonempty values $F(x) \subset Y$ for $x \in X$. A map $F : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(F) = \{(x,y) : x \in X,\ y \in F(x)\}$ is closed in $X \times Y$; *Upper Semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed; *Lower Semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $F^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. If F is u.s.c. with closed values, then F is closed. The converse is true whenever Y is compact.

The concepts of upper or lower semicontinuity of extended real-valued functions are standard. The following is well known [2].

BERGE'S THEOREM. *Let X and Y be topological spaces, $f : X \times Y \to \overline{\mathbf{R}}$ an extended real-valued function, $F : X \multimap Y$ a multimap, and*

$$\hat{f}(x) = \sup_{y \in F(x)} f(x,y), \quad G(x) = \left\{ y \in F(x) : f(x,y) = \hat{f}(x) \right\}, \qquad \text{for } x \in X.$$

(a) *If f is u.s.c. and F is u.s.c. with compact values, then $\hat{f}$ is u.s.c.*
(b) *If f is l.s.c. and F is l.s.c, then $\hat{f}$ is l.s.c.*
(c) *If f is continuous and F is continuous with compact values, then $\hat{f}$ is continuous and G is u.s.c.*

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A *polyhedron* is a set in $\mathbf{R}^n$ homeomorphic to a union of a finite number of compact convex sets in $\mathbf{R}^n$. The product of two polyhedra is a polyhedron [1].

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish. The product of two acyclic spaces is acyclic by the Künneth theorem.

The following is due to Eilenberg and Montgomery [3] or, more generally, to Begle [4].

LEMMA. *Let Z be an acyclic polyhedron and $T : Z \multimap Z$ an acyclic map (that is, u.s.c. with acyclic values). Then T has a fixed point $\hat{x} \in Z$; that is, $\hat{x} \in T(\hat{x})$.*

2. MAIN RESULTS

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j\in I} X_j \quad \text{and} \quad X^i = \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I\setminus\{i\}$, let x^i_j denote the j^{th} coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows. Its i^{th} coordinate is x_i and, for $j \neq i$, its j^{th} coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A\left(x^i\right) = \left\{y_i \in X_i : \left[x^i, y_i\right] \in A\right\} \quad \text{and} \quad A\left(x_i\right) = \left\{y^i \in X^i : \left[y^i, x_i\right] \in A\right\}.$$

From now on, assume that $I = \{1, 2, \ldots, n\}$.

The following is a collectively fixed-point theorem equivalent to the lemma.

THEOREM 1. *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra, and $T_i : X \multimap X_i$ an acyclic map for each $i \in I$. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

PROOF. Note that X itself is an acyclic polyhedron. Define $T : X \multimap X$ by $T(x) = \Pi_{i\in I} T_i(x)$ for each $x \in X$. Then T is a acyclic map. In fact, each T_i is u.s.c. for each $i \in I$ and hence T is also u.s.c.; see [5, Lemma 3]. Note that each $T(x)$ is acyclic. Therefore, by the lemma, T has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T(\hat{x})$, and hence, $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$. This completes our proof.

From Theorem 1, we have the following version of the social equilibrium existence theorem of Debreu [1].

THEOREM 2. *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra, $A_i : X^i \multimap X_i$ closed maps, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. functions for each $i \in I$ such that*

(1) *$g_i(x) \leq f_i(x)$, for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x^i) = \max_{y\in A_i(x^i)} g_i(x^i, y)$ is a l.s.c. function of $x^i \in X^i$; and*
(3) *for each $i \in I$ and $x^i \in X^i$, the set*

$$M\left(x^i\right) = \left\{x_i \in A_i\left(x^i\right) : f_i\left(x^i, x_i\right) \geq \varphi_i\left(x^i\right)\right\}$$

is acyclic.

Then there exists an equilibrium point $\hat{a} \in \mathrm{Gr}(A_i)$, for all $i \in I$; that is,

$$\hat{a}_i \in A_i\left(\hat{a}^i\right) \quad \text{and} \quad f_i\left(\hat{a}\right) = \max_{a_i\in A(\hat{a}^i)} g_i\left(\hat{a}^i, a_i\right), \qquad \text{for all } i \in I.$$

PROOF. For each $i \in I$, define a map $T_i : X \multimap X_i$ by

$$T_i(x) = \left\{y \in A_i\left(x^i\right) : f_i\left(x^i, y\right) \geq \varphi_i\left(x^i\right)\right\}$$

for $x \in X$. Then, $T_i(x) \neq \emptyset$ by (1) since $A_i(x^i)$ is compact and $g_i(x^i, \cdot)$ is u.s.c. on $A_i(x^i)$. We show that $\mathrm{Gr}(T_i)$ is closed in $X \times X_i$. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T_i)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then,

$$\begin{aligned} f_i\left(x^i, y\right) &\geq \overline{\lim_\alpha} f_i\left(x^i_\alpha, y_\alpha\right) \geq \overline{\lim_\alpha} \varphi_i\left(x^i_\alpha\right) \\ &\geq \underline{\lim_\alpha} \varphi_i\left(x^i_\alpha\right) \geq \varphi_i\left(x^i\right), \end{aligned}$$

and since $\mathrm{Gr}(A_i)$ is closed in $X^i \times X_i$, $y_\alpha \in A_i(x^i_\alpha)$ implies $y \in A_i(x^i)$. Hence, $(x, y) \in \mathrm{Gr}(T_i)$. Moreover, each $T_i(x) = M(x^i)$ is acyclic by (3). Now we apply Theorem 1. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$, for all $i \in I$; that is, $\hat{x}_i \in A_i(\hat{x}^i)$ and $f_i(\hat{x}^i, \hat{x}_i) \geq \varphi_i(\hat{x}^i)$. This completes our proof.

Remark 1. If X_i and $M(x^i)$ in (3) are contractible, if $f_i = g_i$ is continuous, and if φ_i is continuous for each $i \in I$, then Theorem 2 reduces to Debreu [1, Theorem]. Note that our proof is much simpler than his.

Remark 2. Since A_i and g_i are u.s.c., by Berge's theorem, φ_i are automatically u.s.c. Hence, condition (2) implies continuity of φ_i.

Remark 3. If A_i and g_i are continuous, condition (2) holds immediately by Berge's theorem, and hence, each φ_i is continuous. This fact is noted by Debreu [1, Remark].

Remark 4. As was also noted by Debreu, instead of acyclic polyhedra, one might take for example absolute retracts or others.

3. APPLICATIONS

From Theorem 2, we obtain acyclic versions of a saddle-point theorem and a minimax theorem.

Corollary 1. *Let X, Y be two acyclic polyhedra, and $f : X \times Y \to \overline{\mathbf{R}}$ a continuous function. Suppose that for each $x_0 \in X$ and $y_0 \in Y$, the sets*

$$\left\{x \in X : f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\right\}$$

and

$$\left\{y \in Y : f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\right\}$$

are acyclic. Then, f has a saddle point $(x_0, y_0) \in X \times Y$; that is,

$$\min_{\eta \in Y} f(x_0, \eta) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0).$$

Proof. Note that a saddle point is a particular case of an equilibrium point for two agents ($n = 2$) in Theorem 2 for $a = (a_1, a_2) = (x, y)$, $X_1 = X$, $X_2 = Y$, $A_1(a^1) = X$, $A_2(a^2) = Y$, $f_1(a) = g_1(a) = f(x, y)$, $f_2(a) = g_2(a) = -f(x, y)$. Note also that condition (2) holds by Berge's theorem.

Corollary 2. *Under the hypothesis of Corollary 1, we have the minimax equality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. By Corollary 1, we have a saddle point $(x_0, y_0) \in X \times Y$ such that

$$\max_{x \in X} f(x, y_0) = f(x_0, y_0) = \min_{y \in Y} f(x_0, y).$$

Since $x \mapsto \min_{y\in Y} f(x,y)$ and $y \mapsto \max_{x\in X} f(x,y)$ are continuous by Berge's theorem and X and Y are compact, they attain maximum on X and minimum on Y, respectively. Therefore,

$$\begin{aligned}\min_{y\in Y}\max_{x\in X} f(x,y) &\le \max_{x\in X} f(x,y_0) = f(x_0,y_0)\\ &= \min_{y\in Y} f(x_0,y) \le \max_{x\in X}\min_{y\in Y} f(x,y).\end{aligned}$$

On the other hand, we clearly have

$$\min_y \max_x f(x,y) \ge \max_x \min_y f(x,y).$$

Therefore, we have the conclusion.

From Theorem 2, we have the following generalization of the Nash equilibrium theorem.

COROLLARY 3. *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra and for each i, $f_i : X \to \overline{\mathbf{R}}$ is a continuous function such that*

(0) *for each $x^i \in X^i$ and each $\alpha \in \overline{\mathbf{R}}$, the set*

$$\left\{x_i \in X_i : f_i\left(x^i, x_i\right) \ge \alpha\right\}$$

is empty or acyclic.

Then there exists a point $\hat{a} \in X$ such that

$$f_i(\hat{a}) = \max_{y_i\in X_i} f_i\left(\hat{a}^i, y_i\right), \qquad \text{for all } i \in I.$$

PROOF. We apply Theorem 2 with $f_i = g_i$ and $A_i : X^i \multimap X_i$ defined by $A_i(x^i) = X_i$ for $x^i \in X^i$. Then, condition (2) of Theorem 2 follows from Berge's theorem, and the set in condition (3) is nonempty and acyclic by (0). Therefore, we have the conclusion.

Finally, note that our results in Section 3 generalize corresponding ones in [1,6–11].

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