PERGAMON

Nonlinear Analysis 33 (1998) 637–644

Nonlinear Analysis

A unified approach to variational inequalities on compact convex sets

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Received 16 May 1995; received in revised form 10 April 1997

Keywords: Multifunction (map); Compact map; U.s.c. map; Convex space; Acyclic; Polytope; Kakutani map; Admissible map; Quasiconcave (quasiconvex); Variational inequality; Graph

1. Introduction

Recently, there have appeared a lot of variations and extensions of variational inequalities and their applications. In the present paper, we show that many of the existence theorems for variational inequality problems are simple consequences of a far-reaching generalization of the Ky Fan minimax inequality due to the first author [19] (Theorem 11).

Many of the known results on variational inequalities are related to the Kakutani maps defined on compact convex subsets of locally convex topological vector spaces. However, our new results are related to very general classes of admissible multimaps including acyclic maps on general topological vector spaces, and subsume many known theorems of the Hartman–Stampacchia–Browder type.

2. Preliminaries

A *multifunction* or *map* $F : X \multimap Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) = \{x \in X : y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{F(x) : x \in A\}$. As usual, F also denotes its graph; that is, $(x, y) \in F$ if and only if $y \in F(x)$. A

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PII: S0362-546X(97)00667-6

map $F : X \multimap Y$ is *compact* provided that the closure $\overline{F(X)}$ of its range is a compact subset of Y. For any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by

$$F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Given two maps $F : X \multimap Y$ and $G : Y \multimap Z$, their *composite* $GF : X \multimap Z$ is defined by $(GF)(x) = G(F(x))$ for $x \in X$.

For topological spaces X and Y, a map $F : X \multimap Y$ is *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, $F^{-}(B)$ is closed in X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, $F^{-}(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

Note that composites of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. See Lassonde [11].

In the sequel, t.v.s. means topological vector spaces. In a t.v.s., a convex hull of any finite subsets will be called a *polytope*.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following properties:

(1) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(2) each $F \in \mathfrak{A}_c$ is u.s.c. and compact valued; and

(3) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the O'Neill maps $\mathbb{N}$ (continuous with values consisting of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ in t.v.s., admissible maps in the sense of Górniewicz, permissible maps of Dzedzej, and others. For details, see [8, 17].

We introduce two more classes:

$F \in \mathfrak{A}_c^\sigma(X, Y) \Leftrightarrow$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X, Y) \Leftrightarrow$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

An approximable map defined by Ben-El-Mechaiekh and Idzik [2] belongs to $\mathfrak{A}_c^\kappa$. Moreover, any u.s.c. compact map defined on a closed subset of a locally convex t.v.s. with closed values is approximable whenever its values are all (1) convex, (2) contractible, (3) decomposable, or (4) ∞-proximally connected. See [2].

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Any class $\mathfrak{A}_c^\kappa$ will be called *admissible*. For details, see [17–19].

Recall that a real-valued function $f : X \to \mathbf{R}$ on a topological space is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is open for each $r \in \mathbf{R}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is convex for each $r \in \mathbf{R}$.

For a t.v.s. E, let E^* denote its topological dual equipped with any topology such that its pairing $\langle\ ,\ \rangle : E^* \times E \to \mathbb{C}$ is continuous. For example, E^* may have the strong topology; that is, the topology of uniform convergence on compact (bounded) subsets of E.

We begin with the following particular form of the first author's [19] (Theorem 11):

Theorem 0. *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X,Y)$ a compact map, and $f,g : X \times Y \to \overline{\mathbf{R}}$ extended real-valued functions. Suppose that*
(1) *$g(x,y) \le f(x,y)$ for each $(x,y) \in X \times Y$;*
(2) *for each $x \in X$, $y \mapsto g(x,y)$ is l.s.c. on Y; and*
(3) *for each $y \in Y$, $x \mapsto f(x,y)$ is quasiconcave on X.*
Then there exists a $\overline{y} \in \overline{F(X)}$ such that

$$\sup_{x\in X} g(x,\overline{y}) \le \sup_{(x,y)\in F} f(x,y).$$

Theorem 0 is a generalization of the Ky Fan minimax inequality [7] and includes a large number of its extensions due to many authors. For details, we refer to [19, 16]. Moreover, Theorem 0 reduces to Chang [5] (Theorem 3.2) whenever $F \in \mathbb{K}(X,Y)$ and $f = g$.

Note that, if F is single valued, then the Hausdorffness assumption on Y is not necessary. See [19].

3. Main results

The following main result extends a number of known existence theorems of solutions of equilibrium problems:

Theorem 1. *Let X be a compact convex space, Z a Hausdorff space, $T \in \mathfrak{A}_c^\kappa(X,Z)$ a map, and $\phi : X \times X \times Z \to \mathbf{R}$ a function. Suppose that*
(1) *$\phi(y,y,z) \le 0$ for each $(y,z) \in X \times Z$;*
(2) *for each $x \in X$, $(y,z) \mapsto \phi(x,y,z)$ is l.s.c. on $X \times Z$; and*
(3) *for each $(y,z) \in X \times Z$, $x \mapsto \phi(x,y,z)$ is quasiconcave on X.*
Then there exist an $\bar{x} \in X$ and a $\bar{z} \in T(\bar{x})$ such that

$$\phi(x,\bar{x},\bar{z}) \le 0 \quad \text{for all} \quad x \in X.$$

Proof. Since X is compact, we may assume that $T \in \mathfrak{A}_c(X,Z)$ and hence T is u.s.c. with compact values. Let $Y := X \times Z$. Define $F : X \multimap Y$ and $f : X \times Y \to \mathbf{R}$ by

$$F(x) := \{x\} \times T(x) \subset Y \quad \text{for} \quad x \in X,$$

$$f(x,(y,z)) := \phi(x,y,z) \quad \text{for} \quad x \in X \quad \text{and} \quad (y,z) \in Y.$$

Then $F \in \mathfrak{A}_c(X,Y)$ and
(4) for each $x \in X$, $(y,z) \mapsto f(x,(y,z))$ is l.s.c. on Y;
(5) for each $(y,z) \in Y$, $x \mapsto f(x,(y,z))$ is quasiconcave on X.

Therefore, by Theorem 0, there exists a $\overline{w} \in \overline{F(X)}$ such that

$$\sup_{x\in X} f(x,\overline{w}) \le \sup_{(x,w)\in F} f(x,w).$$

Since F is u.s.c. with compact values and X is compact, we have $\overline{F(X)} = F(X)$. Hence, $\overline{w} \in F(\bar{x}) = \{\bar{x}\} \times T(\bar{x})$ for some $\bar{x} \in X$, and then, there exists a $\bar{z} \in T(\bar{x})$ such that $\overline{w} = (\bar{x},\bar{z})$. Therefore,

$$\sup_{x\in X} f(x,(\bar{x},\bar{z})) \le \sup_{(x,(x,z))\in F} f(x,(x,z)).$$

However, $f(x,(x,z)) = \phi(x,x,z) \le 0$ by condition (1) implies

$$\sup_{x\in X} \phi(x,\bar{x},\bar{z}) \le 0.$$

This completes our proof. □

Examples. When X is a subset of a t.v.s., there have been many particular forms of Theorem 1 as follows:

1. Fan [6] (Theorem 2): $E = Z$ is a normed vector space, $T = f \in \mathbb{C}(X,E)$, and $\phi(x,y,z) = \|y-z\| - \|x-z\|$ for $(x,y,z) \in X \times X \times E$.
   Fan's theorem can be extended to any $T \in \mathfrak{A}_c^\kappa(X,E)$. Therefore, any $T \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point whenever X is a nonempty compact convex subset of a normed vector space E. This includes the well-known fixed point theorems due to Brouwer, Schauder, Kakutani, Bohnenblust and Karlin, and many others. For further generalizations, we refer to the first author [18].
2. Fan [7] (Corollary 1): $E = Z$, $T : X \hookrightarrow E$ the inclusion, $q : X \times X \to \mathbf{R}$ a continuous function such that for each $y \in X$, $q(y,\cdot)$ is quasiconvex, and $\phi(x,y,z) = q(y,y) - q(y,x)$ for $x,y \in X$ and $z \in E$.
3. Parida and Sen [20] (Theorem 1): $E = \mathbf{R}^n$, Z is a convex subset of $\mathbf{R}^p$, $T \in \mathbb{K}(X,Z)$, and ϕ is continuous.
4. Yang and Chen [24] (Theorem 8): $T = f \in \mathbb{C}(X,E^*)$, $\tau : X \times X \to E$ a function such that, for each $x \in X$, $y \mapsto \tau(x,y)$ is continuous, $\tau(x,x) = 0$, and $y \mapsto \langle f(x),\tau(x,y)\rangle$ is concave; and $\phi(x,y,z) = \langle z,\tau(x,y)\rangle$ for $(x,y,z) \in X \times X \times E^*$.
5. Yao [25] (Lemma): $T = f \in \mathbb{C}(E,E^*)$, $g : E \to E$ is a continuous affine amp, and $\phi(x,y,z) = \langle z, g(x) - g(y)\rangle$ for $(x,y,z) \in X \times X \times E^*$.
6. Parida et al. [21] (Theorem 3.1), Behera and Panda [1] (Theorem 2.2), Siddiqi et al. [23] (Theorem 3.2): $T = f \in \mathbb{C}(X,E^*)$, $\theta : K \times K \to E$ is continuous such that, for each $x \in X$, $\langle T(x),\theta(x,x)\rangle \le 0$ and $y \mapsto \langle T(x),\theta(x,y)\rangle$ is quasiconcave, and $\phi(x,y,z) = \langle z,\theta(x,y)\rangle$ for $(x,y,z) \in X \times X \times E^*$.

From Theorem 1, we obtain the following variational inequality:

Theorem 2. *Let X be a nonempty compact convex subset of a t.v.s. E, E^* have a Hausdorff topology, $T \in \mathfrak{A}_c^\kappa(X,E^*)$ a map, and $\alpha : X \times X \to \mathbf{R}$ a function such that for each $x \in X$, $\alpha(x,x) = 0$, $\alpha(x,\cdot)$ is u.s.c., and $\alpha(\cdot,x)$ is convex.*

Then there exist an $\bar{x} \in X$ *and a* $\bar{z} \in T(\bar{x})$ *such that*

$$\mathrm{Re}\langle \bar{z}, x - \bar{x}\rangle + \alpha(x,\bar{x}) \geq 0 \quad \textit{for all} \quad x \in X.$$

Proof. Let $Z = E^*$ and $\phi : X \times X \times E^* \to \mathbf{R}$ be given by

$$\phi(x,y,z) := \mathrm{Re}\langle z, y - x\rangle - \alpha(x,y)$$

for $(x,y,z) \in X \times X \times E^*$. Then
(1) $\phi(y,y,z) = \alpha(y,y) = 0$ for all $(y,z) \in X \times E^*$;
(2) for each $x \in X$, $(y,z) \mapsto \phi(x,y,z)$ is l.s.c. on $X \times E^*$; and
(3) for each $(y,z) \in X \times E^*$, $x \mapsto \phi(x,y,z)$ is concave on X.
Therefore, by Theorem 1, there exist an $\bar{x} \in X$ and a $\bar{z} \in T(\bar{x})$ such that

$$\mathrm{Re}\langle \bar{z}, \bar{x} - x\rangle - \alpha(x,\bar{x}) \leq 0 \quad \text{for all} \quad x \in X,$$

that is,

$$\mathrm{Re}\langle \bar{z}, x - \bar{x}\rangle + \alpha(x,\bar{x}) \geq 0 \quad \text{for all} \quad x \in X.$$

This completes our proof. □

Examples. 1. Hartman and Stampacchia [10] (Lemma 3.1): $E = E^* = \mathbf{R}^n$, $T \in \mathbb{C}(X,E)$, $\alpha \equiv 0$, and $\langle\ ,\ \rangle$ is the scalar product of $\mathbf{R}^n$.

2. Browder [3] (Theorem 3), [4] (Theorem 2): $T \in \mathbb{C}(X,E^*)$ and $\alpha \equiv 0$.

3. Browder [4] (Theorem 6): $T \in \mathbb{K}(X,E^*)$ and $\alpha \equiv 0$.

4. Lions and Stampacchia, Stampacchia, Mosco [12] (p.94): E is an inner product space, $T : X \to E^*$ is a constant map, and $a : X \times X \to \mathbf{R}$ is a continuous bilinear form such that

$$\alpha(x,y) = -a(y, y - x) \quad \text{for} \quad (x,y) \in X \times X.$$

5. Saigal [22] (Lemma 4.1): $E = E^* = \mathbf{R}^n$, $T \in \mathbb{V}(X,\mathbf{R}^n)$, and $\alpha \equiv 0$.

6. Guo and Kung [9] (Theorem 3.1): $T \in \mathbb{V}(X,E^*)$ and $\alpha(x,y) \equiv f(x) - f(y)$ where $f : E \to \mathbf{R}$ is a l.s.c. convex function.

Let $X \subset E$ and $x \in E$. The inward set of X at x is defined by

$$I_X(x) := x + \bigcup_{r>0} r(X - x).$$

In fixed point theory, the following strengthened forms of Theorems 1 and 2 were shown to be very useful.

Theorem 3. *Let* X, E, Z, *and* T *are the same as in Theorem* 1. *Suppose that a function* $\phi : E \times X \times Z \to \mathbf{R}$ *satisfies the following*:
(1) $\phi(y,y,z) = 0$ *for each* $(y,z) \in X \times Z$;

(2) *for each* $x \in X$, $(y,z) \mapsto \phi(x,y,z)$ *is l.s.c. on* $X \times Z$; *and*
(3) *for each* $(y,z) \in X \times Z$, $x \mapsto \phi(x,y,z)$ *is l.s.c. and concave.*
Then there exist an $\bar{x} \in X$ *and a* $\bar{z} \in T(\bar{x})$ *such that*

$$\phi(x,\bar{x},\bar{z}) \leq 0 \quad \textit{for all} \quad x \in \overline{I_X(\bar{x})}.$$

Proof. By Theorem 1, there exist an $\bar{x} \in X$ and a $\bar{z} \in T(\bar{x})$ such that

$$\phi(x,\bar{x},\bar{z}) \leq 0 \quad \text{for all} \quad x \in X.$$

In order to prove that the inequality holds for all $x \in \overline{I_X(\bar{x})}$, it suffices to show that it holds for all $x \in I_X(\bar{x})$ since $x \mapsto \phi(x,\bar{x},\bar{z})$ is l.s.c. on X. Let $x \in I_X(\bar{x})\backslash X$. Then $x = \bar{x} + r(y - \bar{x})$ for some $y \in X$ and $r > 1$. We have

$$\frac{1}{r}x + (1 - \frac{1}{r})\bar{x} = y \in X$$

and

$$0 \geq \phi(y,\bar{x},\bar{z}) \geq \frac{1}{r}\phi(x,\bar{x},\bar{z}) + (1 - \frac{1}{r})\phi(\bar{x},\bar{x},\bar{z}) = \frac{1}{r}\phi(x,\bar{x},\bar{z})$$

by Eq. (1). This completes our proof. □

Examples. 1. Park [13] (Theorem 1): $E = Z$, $T : X \hookrightarrow E$ the inclusion, $q : X \times E \to \mathbf{R}$ continuous such that for each $y \in X$, $q(y,\cdot)$ is convex, and $\phi(x,y,z) = q(y,y) - q(y,x)$ for all $(x,y,z) \in E \times X \times E$.

2. Park and Kim [14] (Theorem 10): $E = Z$ is a normed vector space, $f \in \mathbb{C}(X,E)$, and $\phi(x,y,z) = \|y - z\| - \|x - z\|$ for $(x,y,z) \in E \times X \times E$.

3. Park and Kim [14] (Corollary 5): $E = Z = \mathbf{R}^n$, $T \in \mathbb{K}(X,\mathbf{R}^n)$, and $\phi(x,y,z) = \langle z - y, x - y\rangle$ for $x,y,z \in \mathbf{R}^n$. This was used to obtain the Kakutani fixed point theorem.

From Theorems 2 or 3, we immediately have the following:

Theorem 4. *Let* X *be a nonempty compact convex subset of a t.v.s.* E, E^* *have the Hausdorff topology,* $T \in \mathfrak{A}_c^\kappa(X,E^*)$ *a map, and* $\alpha : E \times X \to \mathbf{R}$ *an u.s.c. function such that, for each* $x \in X$, $\alpha(x,x) = 0$ *and* $\alpha(\cdot,x)$ *is convex.*

Then there exist an $\bar{x} \in X$ *and a* $\bar{z} \in T(\bar{x})$ *such that*

$$\mathrm{Re}\langle \bar{z}, x - \bar{x}\rangle + \alpha(x,\bar{x}) \geq 0 \quad \textit{for all} \quad x \in \overline{I_X(\bar{x})}.$$

Examples. 1. Park [13] (Theorem 2): $T = f \in \mathbb{C}(X,E^*)$ and $\alpha \equiv 0$. This result was used to establish fixed point theorems for weakly inward maps.

2. Park and Kim [14] (Corollary 4): $T \in \mathbb{K}(X,E^*)$ and $\alpha \equiv 0$.

3. Park [15] (Corollary 2.4): $\alpha \equiv 0$ and $T : X \multimap E^*$ satisfies

(1) $T(x)$ is nonempty convex for each $x \in X$; and

(2) $T^-(y)$ is open for each $z \in E^*$.

Note that $T \in \mathbb{C}_c^\kappa(X,E^*)$.

4. Some of the examples of Theorems 1 and 2 also can be strengthened in the same way.

Acknowledgements

This research was initiated during the first author's visit to Institute of Mathematics, Academia Sinica at Taipei in February 1995. The first author was partially supported by Ministry of Education, Korea, 1997, Project Number BSRI-97-1413, and the second by the National Science Council of the Republic of China under Grant NSC-86-2115-M-001-001.

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