

Approximation and Fixed-Point Theorems for Condensing Composites of Multifunctions

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One of the authors previously extended an interesting (best approximation) theory of Ky Fan to condensing maps defined on a closed ball in a Banach space and defined on a closed convex subset in a Hilbert space. Recently there have appeared a number of new results on fixed points of so-called Kakutani factorizable multifunctions or composites of acyclic or other type of multifunctions defined on convex subsets of topological vector spaces. In this paper, we extend Lin's results to the multifunction (or set-valued) version for the above-mentioned factorizable multifunctions. Applications to fixed-point theorems are given. © 1998 Academic Press

1. INTRODUCTION AND PRELIMINARIES

In 1969, Fan [6] proved the following interesting theorem.

Let K be a nonempty compact convex set in a normed linear space X , and f be a continuous map from K into X . Then there exists $u \in K$ such that

$$\|u - f(u)\| = d(f(u), K).$$

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Since then, various aspects of this theorem have been studied by Fan [7], Lin [13–16], Lin and Yen [17], Reich [24], etc., for the single-valued function case; and by Ha [9], Kong and Ding [10], Park et al. [22], Reich [23], and Sehgal and Singh [25, 26] for multivalued function case.

Recently there have appeared a number of new results on fixed points of so-called Kakutani factorizable multifunctions or composites of acyclic or other type of multifunctions defined on convex subsets of topological vector spaces. For example, see [1, 2, 11, 12, 19, 20–22].

In [13], Lin proved Fan's theorem for a continuous condensing map defined on a closed ball in a Banach space or on a closed convex subset of a Hilbert space. In this paper, we extend Lin's results to the multifunction (or set-valued) version for the above-mentioned factorizable multifunctions.

We first introduce our notations and definitions.

Let B be a nonempty bounded subset of a Banach space X , and $\gamma(B)$ be the (Kuratowski) measure of noncompactness, i.e., let $\gamma(B)$ be the infimum of $\{r > 0 \mid B \text{ can be covered by a finite number of sets of diameter } \leq r\}$ or $\{r > 0 \mid B \text{ can be covered by a finite number of balls of radius } r\}$.

Let S be a nonempty subset of a Banach space X and f be a continuous map from S into X . If for every nonempty bounded subset B of S with $\gamma(B) > 0$, we have $\gamma(f(B)) < \gamma(B)$, then f is called *condensing*. If there exists k , $0 \leq k < 1$, such that for each nonempty bounded subset B of S we have $\gamma(f(B)) \leq k\gamma(B)$, then f is called *k-set-contractive*.

The inward set of S at $x \in S$ is defined by

$$I_S(x) = \{x + c(z - x) \mid z \in S, c > 0\}$$

and $\text{cl } I_S(x)$ is the closure of $I_S(x)$.

Let S be a nonempty subset of a normed linear space X . For each x in X , define

$$d(x, S) = \inf\{\|x - y\| \mid y \in S\},$$

and

$$p_S(x) = \{y \in S \mid \|x - y\| = d(x, S)\}.$$

The set-valued map p_S is called the *metric projection on S* . If P_S is a single-valued map, it is called a *proximity map*. When there is no confusion, we will use p instead of p_S . We will denote the closure of S as $\text{cl } S$, the closed convex hull of S as $\text{cl co } S$, the boundary of S as $\text{bdy } S$, and the interior of S as $\text{int } S$.

A multifunction or map $F: X \rightarrow 2^Y \setminus \{\emptyset\}$ is a function from a set X into the power set of Y , that is, a function with nonempty values $F(x) \subset Y$.

For $A \subset X$, let $F(A) = \cup\{F(x) \mid x \in A\}$. A map $F: X \rightarrow 2^Y$ is compact if $F(X)$ is relatively compact in a topological space Y . For any $B \subset Y$, the (lower) inverse of B under F is defined by $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. Given two maps $F: X \rightarrow 2^Y$ and $G: Y \rightarrow 2^Z$, the composite $GF: X \rightarrow 2^Z$ is defined by $(GF)(x) = G(F(x))$ for each $x \in X$.

For topological spaces X and Y , a map $F: X \rightarrow 2^Y$ is upper semicontinuous (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X ; lower semicontinuous (l.s.c.) if, for each open set $B \subset Y$, $F^-(B)$ is open in X ; and continuous if F is both u.s.c. and l.s.c. A set $K \subset X$ is called σ -compact if K is a countable union of compact sets. A nonempty topological space is acyclic if all of its reduced Čech homology groups over rationals vanish.

In a topological vector space E , any convex hulls of its finite subsets will be called *polytopes*.

Given a class \mathbb{X} of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F: X \rightarrow 2^Y$ belonging to \mathbb{X} , and \mathbb{X}_c the set of finite composites of maps in \mathbb{X} .

A class \mathbb{U} of maps is defined by the following properties:

- (i) \mathbb{U} contains the class \mathbb{C} of (single-valued) continuous functions.
- (ii) Each $F \in \mathbb{U}_c$ is u.s.c. and compact-valued.
- (iii) For any polytope P , $F \in \mathbb{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each \mathbb{U} .

Examples of \mathbb{U} are \mathbb{C} , the Kakutani map \mathbb{K} (with convex values), the Aronszajn maps \mathbb{M} (with R_δ values), the approachable maps \mathbb{A} [1, 2], the acyclic maps \mathbb{V} (with acyclic values), the O'Neill maps \mathbb{N} (continuous with values consisting of one or m acyclic components, with m fixed), admissible maps in the sense of Gorniewicz, permissible maps of Dzedzej, and many others. For details, see [8, 19, 20].

We introduce two more classes:

- (1) $F \in \mathbb{U}_c^\sigma(X, Y)$, if for any σ -compact subset K of X , there is an $\tilde{F} \in \mathbb{U}_c(K, Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.
- (2) $F \in \mathbb{U}_c^\kappa(X, Y)$, if for any compact subset K of X , there is an $\tilde{F} \in \mathbb{U}_c(K, Y)$ such that $\tilde{F}(x) \subset F(x)$ for each $x \in K$.

Note that $\mathbb{U} \subset \mathbb{U}_c \subset \mathbb{U}_c^\sigma \subset \mathbb{U}_c^\kappa$. Those classes are due to one of the authors (see [19, 20]). Examples of \mathbb{U}_c^σ are \mathbb{K}_c^σ due to Lassonde [11] and \mathbb{V}_c^σ due to Park et al. [22]. Moreover, approximable maps due to Ben-El-Mechaiekh and Idzik [3] belong to \mathbb{U}_c^κ . An u.s.c. map defined on a compact convex subset of a locally convex Hausdorff topological vector space with compact values is approximable whenever its values are all convex, contractible, decomposable, or ∞ -proximally connected. See [3].

2. MAIN RESULTS

We need the following particular form of Park [21, Theorem 1]:

LEMMA. *Let S be a closed convex subset of a Banach space X . If $F \in \mathbb{U}_c^\kappa(S, X)$ is a condensing map such that $F(S)$ is bounded and $F(x) \subset I_S(x)$ for all $x \in \text{bdy } S$, then F has a fixed point.*

If X is real and F is a Kakutani map, then the lemma holds whenever $F(x) \cap \text{cl } I_S(x) \neq \emptyset$ for all $x \in \text{bdy } S$. See Deimling [5, Theorem 1].

THEOREM 1. *Let B be a closed ball with center at origin and radius r in a Banach space X . If $F \in \mathbb{U}_c^\kappa(B, X)$ is condensing, then there exist an $x_0 \in B$ and $y_0 \in F(x_0)$ such that*

$$\|x_0 - y_0\| = d(y_0, B) = d(y_0, \text{cl } I_B(x_0)).$$

More precisely, either F has a fixed point $x_0 \in B$; or there exist an $x_0 \in \text{bdy } B$ and $y_0 \in F(x_0)$ such that

$$0 < \|x_0 - y_0\| = d(y_0, B) = d(y_0, \text{cl } I_B(x_0)).$$

Proof. Let $p: X \rightarrow B$ be the radial projection, i.e.,

$$p(x) = \begin{cases} x, & \text{if } \|x\| \leq r \\ rx/\|x\|, & \text{if } \|x\| > r. \end{cases}$$

From [18], p is a continuous 1-set-contractive map. Since F is condensing, it is clear that $p \circ F$ is condensing. It is also clear that \mathbb{U}_c^κ is closed under composition; we have $p \circ F \in \mathbb{U}_c^\kappa(B, B)$. From the lemma, $p \circ F$ has a fixed point x_0 in B ; that is, $x_0 = p(y_0)$ for some $y_0 \in F(x_0)$. We consider the following two cases:

Case 1. If $\|y_0\| \leq r$, then $p(y_0) = y_0 = x_0$, and hence $\|x_0 - y_0\| = 0$. Therefore x_0 is a fixed point of F .

Case 2. If $\|y_0\| \geq r$, then $x_0 = p(y_0) = ry_0/\|y_0\|$, and hence

$$\|x_0 - y_0\| = \|(ry_0/\|y_0\|) - y_0\| = \|y_0\| - r.$$

For each $x \in B$, we have

$$\|y_0\| - r \leq \|y_0\| - \|x\| \leq \|x - y_0\|.$$

Therefore, we have

$$\|x_0 - y_0\| = \mathbf{Min}_{x \in B} \|x - y_0\| = d(y_0, B).$$

Now we show that $\|x_0 - y_0\| = d(y_0, \text{cl } I_B(x_0))$. In fact, for any $y \in I_B(x_0) \setminus B$, there exist $x \in B$ and $r > 0$ such that

$$y = x_0 + r(x - x_0).$$

Then we must have $r > 1$, for otherwise, $y = (1 - r)x_0 + rx \in B$, since $0 < r \leq 1$ and B is convex. Suppose $\|y - y_0\| < \|x_0 - y_0\|$. Since $r > 1$ and

$$(1/r)y + (1 - (1/r))x_0 = x \in B,$$

we have

$$\begin{aligned} \|x - y_0\| &= \|(1/r)(y - y_0) + (1 - (1/r))(x_0 - y_0)\| \\ &\leq (1/r)\|y - y_0\| + (1 - (1/r))\|x_0 - y_0\| < \|x_0 - y_0\|, \end{aligned}$$

which contradicts $\|x_0 - y_0\| = d(y_0, B)$. Therefore we must have

$$\|x_0 - y_0\| \leq \|y - y_0\| \quad \text{for all } y \in I_B(x_0),$$

and, by taking limits on both sides, we have

$$\|x_0 - y_0\| \leq \|y - y_0\| \quad \text{for all } y \in \text{cl } I_B(x_0).$$

Therefore we have

$$\|x_0 - y_0\| = d(y_0, B) = d(y_0, \text{cl } I_B(x_0)).$$

If $x_0 \in \text{int } B$, it is well known that $\text{cl } I_B(x_0) = X$ and $d(y_0, I_B(x_0)) = 0$. This completes the proof. ■

As mentioned in the definitions, \mathbb{U} includes all of the known class of maps. Therefore Theorem 1 is true for a condensing map F that belongs to any of the following class of maps: $\mathbb{K}_c, \mathbb{V}_c, \mathbb{N}_c, \mathbb{K}_c^\kappa, \mathbb{V}_c^\kappa, \mathbb{N}_c^\kappa$, etc. We note that even for the special cases, $F \in \mathbb{K}$ (F is u.s.c. and $F(x)$ is compact convex) or $F \in \mathbb{V}$ (F is u.s.c. and $F(x)$ is compact acyclic), the corresponding result of Theorem 1 has not been proved. In other words, it is new, even for this class of maps. The same comments can be made for Theorem 3.

THEOREM 2. *Let B be a closed ball with its center at the origin and a radius r in a Banach space X . Suppose that $F \in \mathbb{U}_c^\kappa(B, X)$ is condensing and*

satisfies any one of the following conditions for all $x \in \text{bdy } B \setminus F(x)$:

(i) For each $y \in F(x)$, we have $\|y - x\| > \|y - z\|$ for some $z \in \text{cl } I_B(x)$.

(ii) For each $y \in F(x)$, there exists λ , $|\lambda| < 1$, such that $\lambda x + (1 - \lambda)y \in \text{cl } I_B(x)$.

(iii) $F(x) \subset \text{cl } I_B(x)$.

Then F has a fixed point in B .

Proof. By Theorem 1, we have either

(1) F has a fixed point in B ; or

(2) there exist $x_0 \in \text{bdy } B$, and $y_0 \in F(x_0)$ such that

$$0 < \|x_0 - y_0\| = d(y_0, B) = d(y_0, \text{cl } I_B(x_0)).$$

If F satisfies condition (i), it is clear to see that it contradicts (2). Therefore (1) must be true, i.e., F has a fixed point in B .

If F satisfies condition (ii), then F does not satisfy (2). In fact, if F satisfies (2), for the x_0 and y_0 , by condition (ii), there exists λ , $|\lambda| < 1$, such that $\lambda x_0 + (1 - \lambda)y_0 \in \text{cl } I_B(x_0)$, and by (2),

$$\begin{aligned} 0 < \|x_0 - y_0\| &\leq \|y_0 - (\lambda x_0 + (1 - \lambda)y_0)\| \\ &= \|\lambda y_0 + (1 - \lambda)y_0 - (\lambda x_0 + (1 - \lambda)y_0)\| \\ &= |\lambda| \|y_0 - x_0\| < \|y_0 - x_0\|, \end{aligned}$$

which is a contradiction. Hence F does not satisfy (2), and (1) must be true, i.e., F has a fixed point in B .

If F satisfies condition (iii), then F satisfies condition (ii), by taking $\lambda = 0$. ■

THEOREM 3. *Let S be a closed convex subset in a Hilbert space X . If $F \in \mathbb{U}_c^\kappa(S, X)$ is condensing, and $F(S)$ is bounded, then there exists an $x_0 \in S$ and $y_0 \in F(x_0)$ such that*

$$\|x_0 - y_0\| = d(y_0, S) = d(y_0, \text{cl } I_S(x_0)).$$

More precisely, either F has a fixed point $x_0 \in S$, or there exists an $x_0 \in \text{bdy } S$ and $y_0 \in F(x_0)$ such that

$$0 < \|x_0 - y_0\| = d(y_0, S) = d(y_0, \text{cl } I_S(x_0)).$$

Proof. Let $p: X \rightarrow S$ be the metric projection; that is, for each $y \in X$, we have

$$\|p(y) - y\| = d(y, S).$$

From [4], p is nonexpansive. Then $p \circ F$ is condensing. Furthermore, it is clear that $p \circ F \in U_c^k(S, S)$. By the lemma, $p \circ F$ has a fixed point $x_0 \in S$; that is, $x_0 = p(y_0)$ for some $y_0 \in F(x_0)$. Therefore,

$$\|x_0 - y_0\| = \|p(y_0) - y_0\| = d(y_0, S).$$

Following the same proof as in Theorem 1, we can get

$$\|x_0 - y_0\| = d(y_0, \text{cl } I_S(x_0)). \quad \blacksquare$$

THEOREM 4. *Let S be a closed convex subset in a Hilbert space X . If $F \in \mathcal{U}_c^k(S, X)$ is condensing, $F(S)$ is bounded, and F satisfies any one of three conditions of Theorem 2 with $S = B$, then F has a fixed point in S .*

Proof. The proof is similar to that for Theorem 2, except that we use Theorem 3 instead of Theorem 1. \blacksquare

As we noted earlier, if F is single-valued, Theorems 1 and 3 reduce to the results of Lin [13]. Moreover, Theorems 2 and 4 contain some known results. For the literature, see Park [21].

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