LERAY-SCHAUDER TYPE THEOREMS AND EQUILIBRIUM EXISTENCE THEOREMS*

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Abstract. We consider two applications of a fixed point theorem due to the first author. First, we extend the Leray-Schauder theorem to topological vector spaces which are not necessarilly locally convex. As an application we derive some well-known fixed point theorems. Second, we deduce a variation of the social equilibrium existence theorem of Debreu. This is applied to results on saddle points, minimax theorems, and the Nash equilibria.

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1. Introduction and Preliminaries

In this paper, we study two main applications of a fixed point theorem due to the first author [I] related to convexly totally bounded sets. We first extend the Leray-Schauder theorem to topological vector spaces which are not necessarily locally convex. This new result can be used to derive some new or well-known fixed point theorems. Secondly, we deduce a variation of social equilibrium existence theorem of Debreu [D]. This is applied to results on saddle points, minimax theorems, and the Nash equilibria.

All topological vector spaces in this paper are assumed to be real Hausdorff spaces. Given a set X, $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X. In what follows, X and Y are two subsets of two topological vector spaces E and F, respectively. The boundary, the closure, the interior, and the convex hull of a subset X of E are denoted by $\partial X, \overline{X}$, $\operatorname{Int} X$, and $\operatorname{co} X$, respectively. For brevity, locally convex topological vector spaces are called locally convex spaces.

Definitions 1. Let $T : X \to \mathcal{P}(Y)$ be a map.

(1) T is said to be *upper semicontinuous* (u.s.c.) on X if the set $\{x \in X \,|\, T(x) \subset V\}$ is open in X whenever V is an open subset of Y.

(2) T is said to be *compact* if $T(X)$ is relatively compact in Y.

(3) A set $K \subset E$ is *convexly totally bounded* (c.t.b. for short), if for every neighborhood V of $0 \in E$ there exist a finite subset $\{x_i \,|\, i \in I\} \subset K$ ($|I| < +\infty$) and a finite family of convex sets $\{C_i \,|\, i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $K \subset \bigcup_{i \in I}(x_i + C_i)$.

Weber [W] gave another definition of the c.t.b. set:

(4) A set $K \subset E$ is *convexly totally bounded* (c.t.b. for short), if for every neighborhood V of $0 \in E$ there exist a finite subset $\{x_i \,|\, i \in I\} \subset E$ ($|I| < +\infty$) and a finite family of convex sets $\{C_i \,|\, i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $K \subset \bigcup_{i \in I}(x_i + C_i)$.

Observe that

Proposition 1. *The definitions (3) and (4) are equivalent.*

Proof. Let U, V be symmetric neighborhoods of 0 in E such that $U + U \subset V$. We can find $\{x_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$ ($|I| < +\infty$) such that $K \subset \bigcup_{i\in I}(x_i+C_i)$, $x_i \in E$, and $C_i \subset U$ is convex for $i \in I$. Let $y_i \in K\cap(x_i+C_i)$. Then $K \subset \bigcup_{i\in I}(y_i+(x_i-y_i+C_i))$ and the convex set $x_i - y_i + C_i \subset -C_i + C_i \subset U + U \subset V$ for all $i \in I$. $\square$

Proposition 2. *If a compact set K is c.t.b., then the compact set $[0,1]K$ is also c.t.b.*

Proof. Let V be a closed and circled neighborhood of $0 \in E$. By definition $K \subset \bigcup_{i\in I}(y_i + (x_i - y_i + C_i))$ for some $x_i \in E$ and $C_i \subset V$, $i \in I$ ($|I| < +\infty$). Observe that for each $r \in [0,1]$ and $i \in I$, $r(x_i+C_i)\cap K \subset r(x_i+C_i) \subset r(x_i+V) \subset rx_i+V$. Furthermore, we have $[0,1]K = \bigcup_{i\in I}\tilde{C}_i$, where $\tilde{C}_i = \bigcup_{r\in[0,1]} r(x_i + C_i) \cap K$ is a compact set. Thus $\tilde{C}_i$ is covered by a finite number of sets of the form $rx_i + V$, and hence $\tilde{C}_i \subset \bigcup_{j\in J}(r_j x_i + V)$ where $r_j \in [0,1]$, $|J| < +\infty$, $i \in I$. $\square$

Proposition 3. *Every compact subset of a c.t.b. set is c.t.b.* $\square$

The following theorem is a special case of Theorem 4.3 in [I]:

Theorem 1. *Let X be a nonempty convex subset of a t.v.s. E and $T : X \to \mathcal{P}(X)$ be an u.s.c. map with closed convex values. If $\overline{T(X)}$ is a compact c.t.b. subset of X, then T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.* $\square$

2. The Leray-Schauder type theorems

From Theorem 1, we deduce the following Leray-Schauder type theorem:

Theorem 2. *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $T : X \to \mathcal{P}(E)$ be a compact u.s.c. map with closed convex values. If $\overline{T(X)}$ is a c.t.b. subset of E, then either*

(1) *T has a fixed point; or*

(2) *$\lambda x \in T(x)$ for some $\lambda > 1$ and $x \in \partial X$.*

Proof. Let $Y \subset X$ be defined by $Y = \{x \in X \mid x \in tT(x) \text{ for some } t \in [0,1]\}$. Y is nonempty since $0 \in Y$. Moreover, it is closed since T is u.s.c. and has closed values. Therefore, Y is compact since T is compact. Suppose that

(LS) $T(y) \cap \{\lambda y : \lambda > 1\} = \emptyset$ for all $y \in \partial X$.

Then $Y \cap \partial X = \emptyset$. Since X is completely regular, there exists a continuous function $r : X \to [0,1]$ such that $r(x) = 1$ for $x \in Y$ and $r(x) = 0$ for $x \in \partial X$.

Let $S : E \to \mathcal{P}(E)$ be defined by

$$S(x) = \begin{cases} r(x)T(x) & \text{if } x \in X, \\ \{0\} & \text{if } x \notin X. \end{cases}$$

Then S is convex-valued. Since T is compact and closed, so is S. Moreover, $\overline{S(E)}$ is a c.t.b. subset of E as a subset of $[0,1]\overline{T(X)}$. Therefore, by Theorem 1, S has a fixed point. Now $x \in S(x)$ implies $x \in Y$ and $r(x) = 1$. Therefore, $x \in X$ and $x \in T(x)$. This completes our proof. $\square$

We recall a few definitions:

Definitions 2. Let $\mathcal{N}$ be the fundamental system of neighborhoods of 0 in E:

(1) a set $K \subset E$ is said to be *locally convex* if for every $x \in K$ and every $V \in \mathcal{N}$, there exists a $U \in \mathcal{N}$ such that $\operatorname{co}((x + U) \cap K) \subset x + V$, and

(2) a set $K \subset E$ is said to be *of Z type* (see [H]) if for every $V \in \mathcal{N}$ there exists a $U \in \mathcal{N}$ such that $\operatorname{co}(U \cap (K - K)) \subset V$.

The following are well-known:

Proposition 4. *In a locally convex space E every subset $K \subset E$ is of Z type and is a locally convex set.* $\square$

Proposition 5. *If $K \subset E$ is a compact set which is locally convex or of Z type, then it is c.t.b.* $\square$

Corollary. *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $T : X \to \mathcal{P}(E)$ be a compact u.s.c. map with closed convex values. Suppose that one of the following holds:*

(i) *E is locally convex,*
(ii) *the set $\overline{T(X)}$ is locally convex,*
(iii) *the set $\overline{T(X)}$ is of Z type,*
(iv) *the set $\overline{T(X)}$ is c.t.b.*

Then the conclusion of Theorem 2 holds.

The case (i) includes many well-known results. Related theorems have been considered recently by Ben-El-Mechaiekh and Idzik [BI], S. Park [P1, P2], and S. Park and J. A. Park [PP].

From Theorem 2 we can obtain a Schaefer type theorem, Birkhoff-Kellog type theorems and a fixed point theorem for non-selfmaps, as in our previous works; see [P1], [P2].

We give only two results as follows:

Theorem 3. *Let E be a t.v.s. and $T : E \to \mathcal{P}(E)$ be a compact u.s.c. map with closed convex values. If $\overline{T(X)}$ is a c.t.b. subset of E, then either*

(1) *T has a fixed point; or*
(2) *the set $\{x \in E : x \in tT(x)$ for some $t \in (0,1)\}$ is not bounded.* $\square$

Theorem 4. *Let X be a closed convex subset of a t.v.s. E and $T : X \to \mathcal{P}(E)$ is an u.s.c. map with closed convex values such that $T(\partial X) \subset X$. If $\overline{T(X)}$ is a compact c.t.b. subset of E, then T has a fixed point.* $\square$

3. Equilibrium existence theorems

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j\in I} X_j \quad \text{and} \quad X^i = \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I\setminus\{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: Its ith coordinate is x_i and, for $j \neq i$, its jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) = \{y_i \in X_i \mid [x^i, y_i] \in A\} \text{ and } A(x_i) = \{y^i \in X^i \mid [y^i, x_i] \in A\}.$$

The following is a collectively fixed point theorem equivalent to Theorem 1:

Theorem 5. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and $T_i : X = \Pi_{i\in I} X_i \to \mathcal{P}(K_i)$ an u.s.c. map with closed convex values. If $K = \Pi_{i\in I} K_i$ is a c.t.b. subset of X, then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

Proof. Define $T : X \to \mathcal{P}(K)$ by $T(x) = \Pi_{i\in I} T_i(x)$ for each $x \in X$. Then T is a compact u.s.c. map with closed convex values. Since $\overline{T(X)} \subset K$ is a compact c.t.b. subset of X, by Theorem 1, T has a fixed point $\hat{x} \in K$; that is, $\hat{x} \in T(\hat{x})$ and $\hat{x}_i \in T_i(\hat{x})$. □

From Theorem 5, we have the following variation of the social equilibrium existence theorem of Debreu [D]:

Theorem 6. *Let $\{X_i\}_{i\in I}$, E_i, K_i be the same as in Theorem 5. Let $A_i : X^i \to \mathcal{P}(K_i)$ be u.s.c. maps with closed values, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. extended real-valued functions for each $i \in I$, where $\mathrm{Gr}(A_i)$ denotes the graph of A_i. Suppose that*

(1) *$g_i(x) \le f_i(x)$ for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x^i) = \max_{y\in A_i(x^i)} g_i(x^i, y)$ is a l.s.c. function of $x^i \in X^i$; and*
(3) *for each $i \in I$ and $x^i \in X^i$, the set*

$$M(x^i) = \{x_i \in A_i(x^i) \mid f_i(x^i, x_i) \ge \varphi_i(x^i)\}$$

is convex.

If $K = \Pi_{i\in I} K_i$ is a c.t.b. subset of X, then there exists an equilibrium point $\hat{a} \in \mathrm{Gr}(A_i)$ for all $i \in I$; that is,

$$\hat{a}_i \in A_i(\hat{a}^i) \text{ and } f_i(\hat{a}) = \max_{a_i\in A(\hat{a}^i)} g_i(\hat{a}^i, a_i) \text{ for all } i \in I.$$

Proof. For each $i \in I$, define a map $T_i : X \to \mathcal{P}(X_i)$ by

$$T_i(x) = \{y \in A_i(x^i) \mid f_i(x^i, y) \ge \varphi_i(x^i)\}$$

for $x \in X$. Then $T_i(x) \ne \emptyset$ by (1) since $A_i(x^i)$ is compact and $g_i(x^i, \cdot)$ is u.s.c. on $A_i(x^i)$. We show that $\mathrm{Gr}(T_i)$ is closed in $X \times X_i$. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T_i)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f_i(x^i, y) &\ge \overline{\lim_\alpha} f_i(x^i_\alpha, y_\alpha) \ge \overline{\lim_\alpha} \varphi_i(x^i_\alpha) \\ &\ge \underline{\lim_\alpha} \varphi_i(x^i_\alpha) \ge \varphi_i(x^i) \end{aligned}$$

and, since $\mathrm{Gr}(A_i)$ is closed in $X^i \times X_i$, $y_\alpha \in A_i(x^i_\alpha)$ implies $y \in A_i(x^i)$. Hence $(x, y) \in \mathrm{Gr}(T_i)$. Therefore, T_i is u.s.c. with convex values $T_i(x) = M(x^i)$ by (3). Now we apply Theorem 5. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$; that is, $\hat{x}_i \in A_i(\hat{x}^i)$ and $f_i(\hat{x}_i, \hat{x}_i) \ge \varphi_i(\hat{x}^i)$. This completes our proof. □

From Theorem 6, we obtain a saddle point theorem and a minimax theorem.

Theorem 7. *Let X, Y be two compact convex c.t.b. subsets, each in a t.v.s., and $f : X \times Y \to \overline{\mathbf{R}}$ a continuous function. Suppose that for each $x_0 \in x$ and $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\}$$

and

$$\{y \in Y : f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are convex. then f has a saddle point $(x_0, y_0) \in X \times Y$; that is,

$$\min_{\eta \in Y} f(x_0, \eta) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0).$$

Proof. Note taht a saddle point is a particular case of an equilibrium point for two agents ($n = 2$) in Theorem 2 for $a = (a_1, a_2)$, $X_1 = X$, $X_2 = Y$, $A_1(a^1) = X$, $A_2(a^2) = Y$, $f_1(a) = g_1(a) = f(x, y)$, $f_2(a) = g_2(a) = -f(x, y)$. Note that condition (2) holds by Berge's theorem [B, Theorem VI.3.2]. □

Theorem 8. *Under the hypothesis of Theorem 7, we have the minimax inequality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. By Theorem 7, we have a saddle point $(x_0, y_0) \in X \times Y$ such that

$$\max_{x \in X} f(x, y_0) = f(x_0, y_0) = \min_{y \in Y} f(x_0, y).$$

Since $x \mapsto \min_{y \in Y} f(x, y)$ and $y \mapsto \max_{x \in X} f(x, y)$ are continuous by Berge's theorem and X and Y are compact, they attain maximum on X and minimum or Y, respectively. Therefore,

$$\begin{aligned} \min_{y \in Y} \max_{x \in X} f(x, y) &\leq \max_{x \in X} f(x, y_0) = f(x_0, y_0) \\ &= \min_{y \in Y} f(x_0, y) \leq \max_{x \in X} \min_{y \in Y} f(x, y). \end{aligned}$$

On the other hand, we clearly have

$$\min_{y} \max_{x} f(x, y) \geq \max_{x} \min_{y} f(x, y).$$

Therefore, we have the conclusion. □

From Theorem 6, we have the following generalization of the Nash equilibrium theorem:

Theorem 9. *Let $\{X_i\}_{i \in I}$ be a family of compact convex sts, each in a t.v.s. E_i and for each i, $f_i : X \to \overline{\mathbf{R}}$ a continuous function such that*

(0) *for each $x^i \in X^i$ and each $\alpha \in \overline{\mathbf{R}}$, the set*

$$\{x_i \in X_i : f_i(x^i, x_i) \geq \alpha\}$$

is empty or convex.

If $X = \Pi_{i \in I} X_i$ is a c.t.b. subset of $E = \Pi_{i \in I} E_i$, then there exists a point $\hat{a} \in X$ such that

$$f_i(\hat{a}) = \max_{y_i \in X_i} f_i(\hat{a}^i, y_i) \text{ for all } i \in I.$$

Proof. We apply Theorem 6 with $f_i = g_i$ and $A_i : X^i \multimap X_i$ defined by $A_i(x^i) = X_i$ for $x^i \in X^i$. Then condition (2) of Theorem 6 follows from Berge's theorem, and the set in condition (3) is nonempty and convex by (0). Therefore, we have the conclusion. □

Finally, note that Theorems 6-8 generalize corresponding results of von Neumann, Kakutani, Nash, and von Neumann and Morgenstern; for the literature, see Debreu [D].

References

[B1] H. Ben-El-Mechaiekh and A. Idzik, *A Leray-Schauder type theorem for approximable maps*, Proc. Amer. Math. Soc. **122** (1994), 105–109. MR 94k:54074.

[B] C. Berge, *Espaces Topologiques*, Dunod, Paris, 1959.

[D] G. Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. USA **38** (1952), 886–893 [= Chap. 2, *Mathematical Economics: Twenty Papers of gerald Debreu*, Cambridge Univ. Press, Cambridge, 1983].

[H] O. Hadžić, *Fixed point theorems in not necessarily locally convex topological vector spaces*, Lecture Notes in Math., **948**, Springer-Verlag, Berlin, 1982, pp.118–130.

[I] A. Idzik, *Almost fixed point theorems*, Proc. Amer. Math. Soc. **104** (1988), 779–784.

[P1] S. Park, *Generalized Leray-Schauder principles for compact admissible multifunctions*, Topol. Meth. in Nonlinear Analysis **5** (1995), 271–277.

[P2] ———, *Fixed points of approximable maps*, Proc. Amer. Math. Soc. **124** (1996), 3109–3114.

[PP] S. Park and J. A. Park, *The Idzik type quasivariational inequalities and noncompact optimization problems*, Colloq. Math. **71** (1996), 287–295.

[W] H. Weber, *Compact convex sets in non-locally convex linear spaces, Schauder-Tychonoff fixed point theorem*, in: Topology, Measure and Fractals (Warnemünde, 1991), ed. C. Bandt et al., Math. Res. **66**, Akademie-Verlag, Berlin, 1992, 37–40.