FIXED POINTS OF σ-SELECTIONABLE MULTIMAPS

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ABSTRACT. The fixed point theorems of Haddad and Lasry [1] are extended to compact σ-selectionable maps defined on an admissible convex subset of a topological vector space E which is not necessarily locally convex and not metrizable.

Key Words and Phrases. Multimap (map), u.s.c. map, closed map, compact map, selectionable, σ-selectionable, polytope, Hausdorff topological vector space (t.v.s.), admissible set (in the sense of Klee).

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1. INTRODUCTION

Haddad and Lasry [1] gave conditions ensuring the existence of periodic solutions in the case of functional differential inclusions with memory and as one of the particular cases for ordinary differential inclusions. Such existence properties were shown to be closely related to the existence of a fixed point for a multimap depending on the initial value problem associated with the considered functional differential inclusion.

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In [1], the abstract notion of σ-selectionable multimaps were defined and some important fixed point theorems were developed for such a class of multimaps from a nonempty compact convex subset H of a locally convex metrizable vector space E into H.

In the present paper, we show that the fixed point theorems in [1] can be easily extended to compact σ-selectionable multimaps defined on a nonempty convex subset H of a topological vector space E which is not necessarily locally convex and not metrizable.

Our method is based on our previous works [5-7].

2. Preliminaries

A *multimap* or *map* $\Gamma : X \multimap Y$ is a function from X into the power set of Y with nonempty values $\Gamma(x)$ for $x \in X$ and fibers $\Gamma^{-}(y) = \{x \in X : y \in \Gamma(x)\}$ for $y \in Y$.

For topological spaces X and Y, a map $\Gamma : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if the set $\Gamma^{+}(\Omega) = \{x \in X : \Gamma(x) \subset \Omega\}$ is open in X for each open subset Ω of Y; *closed* if its graph $\mathrm{Gr}(\Gamma) = \{(x, y) : x \in X,\ y \in \Gamma(x)\}$ is closed in $X \times Y$; and *compact* if the closure $\overline{\Gamma(X)}$ of its range $\Gamma(X)$ is compact in Y.

In this paper, we assume that every u.s.c. map has compact values, and that t.v.s. means Hausdorff topological vector spaces.

A map $\Gamma : X \multimap Y$ is said to be *selectionable* if there exists a continuous map $\gamma : X \to Y$ such that $\gamma(x) \in \Gamma(x)$ for all $x \in X$.

A map $\Gamma : X \multimap Y$ is said to be σ-*selectionable* if there exists a sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ of u.s.c. selectionable maps $\Gamma_n : X \multimap Y$ such that

(a) $\Gamma_{n+1}(x) \subset \Gamma_n(x)$ for any $x \in X$ and any $x \in \mathbb{N}$; and

(b) $\Gamma(x) = \bigcap_{n\in\mathbb{N}} \Gamma_n(x)$ for any $x \in X$.

The sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ will be called a *representative sequence* of Γ.

Haddard and Lasry [1] defined the above notions for metric spaces X and Y, and gave some algebraic results associated with them and a number of important examples.

In fact, the following are some examples of σ-selectionable maps [1]:

(1) Every u.s.c. selectionable map.

(2) Every u.s.c. map with convex values from a metric space X into a compact convex subset H of a locally convex metrizable vector space E.

(3) Every u.s.c. map Γ with convex values from a metric space X into a Fréchet space E whenever Γ is totally bounded (i.e., the image by Γ of any ball in X is relatively compact in E).

(4) Any finite compositions, products (see Lemma 3 below), or linear combinations of the above type of maps.

We need the following:

Lemma 1. [1, Proposition A.I.1] *Every σ-selectionable map $\Gamma : X \multimap Y$ is u.s.c.* (*with nonempty compact values*).

Lemma 2. [1, Proposition A.I.3] *Let $\Gamma : X \multimap Y$ be a σ-selectionable and $f : Y \to Z$ continuous. Then $f \circ \Gamma : X \multimap Z$ is σ-selectionable.*

Lemma 3. [1, Proposition A.I.4] *Let $\Gamma : X \multimap Y$ and $\Delta : X \multimap Z$ be σ-selectionable. The map $\Gamma \times \Delta : X \multimap Y \times Z$ defined by $(\Gamma \times \Delta)(x) = \Gamma(x) \times \Delta(x)$ for $x \in X$ is also σ-selectionable.*

Theorem 4. [1, Theorem A.III.1] *Let E be a locally convex metrizable t.v.s. and K a nonempty compact convex subset of E. Then any σ-selectionable map $\Gamma : K \multimap K$ has a fixed point $x_* \in K$, that is, $x_* \in \Gamma(x_*)$.*

3. Main results

Our main aim in this paper is to show that Theorem 4 holds for a compact σ-selectionable map $\Gamma : K \multimap K$, where K is a nonempty convex subset of a t.v.s. E in a class larger than that of locally convex metrizable ones.

Let X be a convex subset of a t.v.s. E. A (finite dimensional) *polytope* P in X is the convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

Lemma 5. *Let X be a nonempty convex subset of a t.v.s. E, P a polytope in X, and Y a topological space. Let $\Gamma : X \multimap Y$ be σ-selectionable and $f : \Gamma(P) \to P$ a continuous map. Then the composition $f \circ (\Gamma|_P) : P \multimap P$ has a fixed point.*

Proof. Since $\Gamma|_P : P \multimap \Gamma(P)$ is σ-selectionable and $f : \Gamma(P) \to P$ is continuous, by Lemma 2, $f \circ (\Gamma|_P) : P \multimap P$ is σ-selectionable. Note that P is a nonempty compact convex subset of a finite dimensional subspace of E. Therefore, it follows from Theorem 4 that $f \circ (\Gamma|_P)$ has a fixed point.

Remark. For the simplicity, we derived Lemma 5 from Theorem 4. However, we can give a direct proof of Lemma 5 by modifying that of Theorem 4.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [3]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are ℓ^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. or any compact convex locally convex subset of a t.v.s. is admissible. For details, see Hadžić [2], Weber [8], and references therein.

The following is the main result of this paper.

Theorem 6. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact σ-selectionable map $\Gamma : X \multimap X$ has a fixed point.*

Proof. By Lemma 1, Γ is a compact u.s.c. map with closed values, and hence Γ is closed. Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin 0 of E. Since Γ is closed and compact, it is sufficient to show that for any $V \in \mathcal{V}$, there exists an $x_V \in X$ such that $(x_V + V) \cap \Gamma(x_V) \neq \emptyset$.

Since $\overline{\Gamma(X)}$ is a compact subset of the admissible subset X, there exist a continuous map $h : \overline{\Gamma(X)} \to X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{\Gamma(X)}$ and $h(\overline{\Gamma(X)}) \subset L \cap X$. Let $M := h(\overline{\Gamma(X)})$. Then M is a compact subset of L and hence $P := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{\Gamma(X)} \to P$ and $\Gamma|_P : P \multimap \Gamma(P) \subset \overline{\Gamma(X)}$. Hence $(h|_{\Gamma(P)}) \circ (\Gamma|_P) : P \multimap P$. Since Γ is σ-selectionable and $f := h|_{\Gamma(P)} : \Gamma(P) \to P$ is continuous, by Lemma 5, their composition $f \circ (\Gamma|_P) : P \multimap P$ has a fixed point $x_V \in P$; that is, $x_V \in (h \circ \Gamma)(x_V)$ and hence $x_V = h(y)$ for some $y \in \Gamma(x_V) \subset \overline{\Gamma(X)}$. Since $y - h(y) \in V$, we have $y \in h(y) + V = x_V + V$. Therefore, $(x_V + V) \cap \Gamma(x_V) \neq \emptyset$. This completes our proof.

Examples. 1. Theorem 4 [1, Theorem A.III.1] is a particular form of Theorem 6 with respect to several aspects.

2. [1, Corollary A.III.1] is a particular form of Theorem 6 for a Fréchet space E and a closed convex subset X of E.

Remark. Recently the author introduced the class $\mathfrak{B}(X, Y)$ of "better" admissible multimaps $\Gamma : X \multimap Y$ [5]. The basic fixed point theorems for such class of multimaps were given in [5-7]. Lemma 5 shows that $\Gamma \in \mathfrak{B}(X, Y)$ whenever Γ is σ-selectionable and hence Theorem 6 is a consequence of [6, Theorem 1] or [7, Corollary 1.1]. Moreover, note that [1, Corollaries A.III.2-4] are consequences of Theorem 6 and can be extended for the maps in $\mathfrak{B}(H, E)$.

The following collectively fixed point theorem is equivalent to Theorem 6:

Theorem 7. *Let* $\{X_i\}_{i=1}^n$ *be a family of nonempty convex sets, each in a t.v.s.* E_i*,* K_i *a nonempty compact subset of* X_i*, and* $\Gamma_i : X = \Pi_{i=1}^n X_i \multimap K_i$ *a* σ*-selectionable map for each* i*,* $1 \leq i \leq n$*. If* X *is admissible in* $E = \Pi_{i=1}^n E_i$*, then there exists an* $\hat{x} \in K = \Pi_{i=1}^n K_i$ *such that* $\hat{x}_i \in \Gamma_i(\hat{x})$ *for each* i*, where* $\hat{x}_i$ *is the* i*th coordinate of* $\hat{x}$*.*

Proof. Define $\Gamma : X \multimap K$ by $\Gamma(x) = \Pi_{i=1}^n \Gamma_i(x)$. Then Γ is σ-selectionable by Lemma 3. Since K is compact in the admissible convex subset X of E, by Theorem 6, Γ has a fixed point $\hat{x} \in K$; that is, $\hat{x} \in \Gamma(\hat{x})$ such that $\hat{x}_i \in \Gamma_i(\hat{x})$ for each i. This completes our proof.

Remark. It is well-known that Theorem 7 remains valid if each Γ_i is allowed to be Kakutani maps or acyclic maps instead of σ-selectionable maps. In this case, Theorem 7 were known to be very useful in economical equilibrium theory. Finally further information on fixed point theorems on t.v.s. can be seen in [2], [4].

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