REMARKS ON FIXED POINTS OF LOWER SEMICONTINUOUS MAPS

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Abstract. From a selection theorem of Michael [M2] and a recent fixed point theorem due to author [P4], we obtain a Leray-Schauder type theorem and unify some recent results on l.s.c. multimaps with convex values.

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1. Introduction

In 1997, there have appeared at least five papers which were concerned with fixed point theorems for lower semicontinuous multimaps with convex values; see [O, KL, W, Za, Ze]. Most of those theorems are based on one of the well-known selection theorems of Michael, for example, [M2]; and some have applications to the existence of equilibrium points of abstract economies.

On the other hand, the author obtained very general fixed point theorems for compact multimaps in the class $\mathfrak{B}$ of "better" admissible maps defined on admissible convex subsets (in the sense of Klee) of a Hausdorff topological vector space

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(t.v.s.); see [P3,4]. Our aim in this paper is to derive recent fixed point theorems for lower semicontinuous multimaps from one of our theorems.

2. Preliminaries

A *multimap* or *map* $F : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in F^-(y)$ if and only if $y \in F(x)$.

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $F^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is closed in X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $F^-(B)$ is open; *closed* if its graph $\mathrm{Gr}(F) = \{(x, y) : x \in X, y \in F(x)\}$ is closed in $X \times Y$; and *compact* if the closure $\overline{F(X)}$ of its range $F(X) = \bigcup_{x \in X} F(x)$ is compact in Y. If F is u.s.c. with closed values and Y is regular, then F is closed. The converse is true whenever Y is compact.

The following is a well-known selection theorem of Michael [M2, Theorem 1.2]:

Theorem 0. *Let X be a paracompact space and M a metrizable subset of a complete locally convex t.v.s. E. Let $F : X \multimap M$ be a l.s.c. map such that, for some metric on M, every $F(x)$ is complete. Then there exists a continuous $f : X \to E$ such that $f(x) \in \overline{co}\, F(x)$ for every $x \in X$.*

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of an admissible t.v.s. are ℓ^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable function on $[0,1]$, the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others; see [P4] and references therein.

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is any convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

We define the *"better" admissible class* $\mathfrak{B}$ of multimaps defined on X as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that for any polytope P in X and any continuous map $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

Subclasses of $\mathfrak{B}$ are classes of continuous function $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (u.s.c. with closed convex values and codomains are convex), the acyclic maps $\mathbb{V}$ (u.s.c. with compact acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), and many others; see [P3,4].

The following is due to the author [P4]:

Theorem 1. *Let E be a t.v.s. and X an admissible convex subset of E. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Note that Theorem 1 includes a large number of historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Fan and Glicksberg, Himmelberg, and many others; for the references, see [P1].

3. Main Results

We begin with the following Leray-Schauder type fixed point theorem:

Theorem 2. *Let E be a locally convex t.v.s., U a convex neighborhood of the origin 0 of E, and X a convex subset of E containing 0. Then any closed compact map $F \in \mathfrak{B}(\overline{U} \cap X, X)$ satisfying*

(LS) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$ *for each* $x \in \mathrm{Bd}_X U$

has a fixed point.

Proof. Let p be the Minkowski functional of $\overline{U}$. Since $0 \in U$, p is continuous. Define $r : E \to \overline{U}$ by $r(x) = x$ for $x \in \overline{U}$ and $r(x) = p(x)^{-1}x$ for $x \notin \overline{U}$; that is,

$$r(x) = [\max\{1, p(x)\}]^{-1} x \quad \text{for} \quad x \in E.$$

Then r is a continuous retraction of E onto $\overline{U}$. Moreover, since X is convex and $0 \in X$, we have $r(X) \subset X$ and $r(F(\overline{U}) \cap X) \subset \overline{U} \cap X$. Define $G = rF$. Since $F \in \mathfrak{B}(\overline{U} \cap X, X)$, it is easy to see that $G = rF \in \mathfrak{B}(\overline{U} \cap X, \overline{U} \cap X)$ and G is closed and compact. Therefore, by Theorem 1, there exists an $x \in \overline{U} \cap X$ such that $x \in Gx$; that is, $x = r(y)$ for some $y \in Fx$.

Suppose that $x \in \mathrm{Int}\, U$. Then

$$1 > p(x) = p(r(y)) = [\max\{1, p(y)\}]^{-1} p(y).$$

Hence $p(y) < 1$ and this implies $r(y) = y$. Then $x = r(y) = y \in F(x)$.

On the contrary, suppose that $x \in \mathrm{Bd}_X U$. Then

$$1 = p(x) = p(r(y)) = [\max\{1, p(y)\}]^{-1} p(y).$$

If $p(y) < 1$, we have a contradiction. If $p(y) > 1$, then $x = r(y) = p(y)^{-1}y$ and $y = p(y)x$. This contradicts condition (LS). Thus $p(y) = 1$, so that $r(y) = y$ and $x = r(y) = y \in F(x)$. This competes our proof.

Remark. A slightly weaker version of Theorem 2 was given by the author [P2, Theorem 2].

From Theorems 0 and 2, we can deduce the following nonlinear alternative of Leray-Schauder type for l.s.c. multimaps due to O'Regan [O, Theorem 3.1]:

Corollary 3. *Let X be either* (i) *a closed convex subset of a Fréchet space (complete metrizable locally convex t.v.s.) E, or* (ii) *a compact metrizable convex subset of a locally convex t.v.s. E. Let U be a relatively open convex subset of X containing* 0 *and $F : \overline{U} \multimap X$ a l.s.c. compact map with closed convex values. Then either* (a) *F has a fixed point in $\overline{U}$; or*

(b) *there exists an $x_0 \in \operatorname{Bd} U$ and a $\lambda \in (0,1)$ with $u \in \lambda F x_0$.*

Proof. Since $\overline{U}$ is paracompact as a subset of a metrizable set X, by Theorem 0, $F : \overline{U} \multimap X$ has a continuous selection $f : \overline{U} \to X$. Then f is a closed compact map and $f \in \mathbb{C}(\overline{U}, X) = \mathbb{C}(\overline{U} \cap X, X)$. Suppose that (b) does not hold. Then we have

(LS) $\{f(x)\} \cap \{\lambda x : \lambda > 1\} = \emptyset$ for each $x \in \operatorname{Bd} U$.

Therefore, by Theorem 2, f has a fixed point $x_0 \in \overline{U}$; that is, $x_0 = f(x_0) \in F(x_0)$. This completes our proof.

Remark. Note that $F \in \mathfrak{B}(\overline{U}, X)$ in Corollary 3.

From Theorems 0 and 1, we obtain the following result which unifies some of recent works:

Theorem 4. *Let X be a convex subset of a locally convex t.v.s. E, Y a compact metrizable subset of X, and $S : X \multimap Y$ a l.s.c. map with closed convex values. Then S has a fixed point.*

Proof. Since Y is compact, by the well-known argument of Fournier and Granas [FG] (see also Lassonde [L]), $\operatorname{co} Y$ is a σ-compact subset of X and hence $\operatorname{co} Y$ is Lindelöf. Since $\operatorname{co} Y$ is regular as a subset of a t.v.s., we know that $\operatorname{co} Y$ is paracompact. Then, by Theorem 0, there exists a continuous map $f : \operatorname{co} Y \to \overline{E}$, where $\overline{E}$ is the completion of E, such that $f(x) \in S(x) \subset Y \subset \operatorname{co} Y$ for all $x \in \operatorname{co} Y$. Note that $f \in \mathbb{C}(\operatorname{co} Y,\ \operatorname{co} Y)$ and f is compact. Therefore, by Theorem 1, f has a fixed point $x_0 \in \operatorname{co} Y \subset X$; that is, $x_0 = f(x_0) \in T(x_0)$. This completes our proof.

Remarks. 1. Theorem 4 is due to Wu [W, Corollary 3]. Note that [W, Corollary 2] is a simple consequence of Theorem 4. For if S is l.s.c., so is $\overline{\operatorname{co}}\, S$ by Michael [M1, Propositions 2.3 and 2.6].

2. Theorem 4 generalizes Kim and Lee [KL, Corollary to Theorem 3] in several aspects; that is, their result is a particular case of Theorem 4 for $X = Y$ and E itself is metrizable.

3. Zheng [Ze, Theorem 2.5] obtained a particular form of Theorem 4 for a normed vector space E.

4. Zhang [Za, Lemma 2.1] deduced a particular form of Theorem 4 for a Banach space E from a result of Cubioti.

5. Note that $S \in \mathfrak{B}(X, Y)$ in Theorem 4.

Theorem 4 has the following more general form.

Theorem 5. *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a locally convex t.v.s. E_i, Y_i a nonempty compact metrizable subset of X_i, and $S_i : X := \Pi_{i\in I} X_i \multimap Y_i$ a l.s.c. map with closed convex values. Then there exists a point $\overline{x} = \Pi_{i\in I}\overline{x}_i \in Y := \Pi_{i\in I} Y_i$ such that $\overline{x}_i \in S_i(\overline{x})$ for each $i \in I$.*

Proof. Note that $X = \Pi_{i\in I} X_i$ is a convex subset of a locally convex t.v.s. $E := \Pi_{i\in I} E_i$ and that $Y = \Pi_{i\in I} Y_i$ is a compact subset of X. Then, as in the proof of Theorem 4, $\operatorname{co} Y$ is a paracompact subset of X. Then by Theorem 0, for each $i \in I$, there exists a continuous map $f_i : \operatorname{co} Y \to Y_i$ such that $f_i(x) \in S_i(x) \subset Y_i$ for all $x \in \operatorname{co} Y$. Define $f : \operatorname{co} Y \to Y$ by $f(x) = \Pi_{i\in I} f_i(x)$ for $x \in \operatorname{co} Y$. Then $f \in \mathbb{C}(\operatorname{co} Y, \operatorname{co} Y)$ and f is compact. Therefore, by Theorem 1, f has a fixed point $\overline{x} \in Y$; that is, $\overline{x}_i = f_i(\overline{x}) \in S_i(x)$ for each $i \in I$. This completes our proof.

Remarks. 1. Theorem 5 is equivalent to Wu [W, Theorem 1]. Note that our statement and proof are much simpler than his. If each E_i is a normed vector space, Theorem 5 reduces to Zheng [Ze, Corollary 3.1].

2. Wu [W, Theorem 4-7] applied Theorem 5 to obtain new equilibrium existence theorems for asbtract economies.

3. Finally, in view of Theorem 1, the main results of Kim and Lee [KL, Theorems 1 and 2] can be extended to non-locally convex spaces.

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