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An Application of a Browder-Type Fixed Point Theorem to Generalized Variational Inequalities

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The purpose of this paper is to provide an application of a non-compact version, due to Park, of Browder's fixed point theorem to generalized variational inequalities. In a non-compact setting, we establish a fairly general existence theorem on a generalized variational inequality using the result of Park. Our main result is not only a strict generalization of Ding and Tan's result without assuming any continuity of given functions, but also a purely infinite dimensional version of recent results of Yao and Guo. As an immediate consequence, we obtain an extension of a result of Browder concerned with generalized variational inequalities without assuming the local convexity of the underlying topological vector space.

1. INTRODUCTION

In 1968, Browder [3, Theorem 1] established his famous fixed point theorem based upon only two elementary topological tools: the Brouwer fixed point theorem [2] and the partition of unity argument. This fixed point theorem is equivalent to the celebrated Fan's lemma [5] which is an infinite-dimensional generalization of the classical KKM theorem [6]. With

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this result alone, Browder [3] carried through a complete treatment of a wide range of fixed point theory, minimax theory, variational inequalities, monotone operator, and game theory.

The purpose of this paper is to provide an application of a non-compact version due to Park [9, Theorem 1] of Browder's fixed point theorem to generalized variational inequalities (in short, GVI). In a non-compact setting, we establish a fairly general existence theorem on GVI using the result of Park [9, Theorem 1]. This existence theorem [Theorem 1] is designed to unify and to sharpen recent results of Ding and Tan [4, Theorem 4] and Yao and Guo [10, Theorems 4.1 and 4.5] concerned with GVI. Our main result, Theorem 1, is not only a strict generalization of Ding and Tan [4, Theorem 4] without assuming any continuity of given functions, but also a purely infinite dimensional version of Yao and Guo [10, Theorems 4.1 and 4.5]. As an immediate consequence, we obtain an extension of Browder [3, Theorem 6] without assuming the local convexity of the underlying topological vector space. Our basic idea is motivated by Yao and Guo [10].

2. PRELIMINARIES

For the terminologies and notations, we follow mainly [4, 9]. Let X and Y be sets. A *multifunction* $S : X \to 2^Y$ is a function from X into the power set 2^Y of Y. For $y \in Y$, let $S^{-1}y = \{x \in X \mid y \in Sx\}$. A *convex space* X is a convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. In particular, any convex subset X with the relative topology of a real Haussdorf topological vector space (t.v.s.) E is obviously a convex space.

A subset B of a topological space Y is said to be *compactly open* in Y if for every compact set $K \subset Y$ the set $B \cap K$ is open in K. An extended real-valued function $f : X \to R \cup \{+\infty\}$ on a topological space X is lower semicontinuous (l.s.c.) if $\{x \in X \mid fx > r\}$ is open for each $r \in R$.

For topological spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) provided that for each open subset V of Y, we have $\{x \in X \mid \emptyset \neq Fx \subset V\}$ is open in X; and *lower semicontinuous* (l.s.c.) provided that for each open subset V of Y, we have $\{x \in X \mid Fx \cap V \neq \emptyset\}$ is open in X. For a convex space Y, $kc(Y)$ denotes the set of all nonempty compact convex subsets of Y and $\mathscr{F}(Y)$ the set of all nonempty finite subsets of Y.

Let E and F be two real vector spaces, and $\langle\, , \rangle : F \times E \to R$ be a bilinear functional. For each $x_0 \in E$, each nonempty subset A of E and

$\epsilon > 0$, let

$$W(x_0; \epsilon) := \{y \in F : |\langle y, x_0 \rangle| < \epsilon\},$$

$$U(A; \epsilon) := \left\{ y \in F : \sup_{x \in A} |\langle y, x \rangle| < \epsilon \right\}.$$

Let $\sigma\langle F, E\rangle$ be the topology on F generated by the family $\{W(x; \epsilon) \mid x \in E$ and $\epsilon > 0\}$ as a subbase for the neighborhood system at the origin. If E is a t.v.s., let $\delta\langle F, E\rangle$ be the topology on F generated by the family $\{U(A; \epsilon) \mid A$ is a nonempty compact subset of E and $\epsilon > 0\}$ as a base for the neighborhood system at the origin. We note then, F when equipped with the topology $\sigma\langle F, E\rangle$ or the topology $\sigma\langle F, E\rangle$ becomes a locally convex space but not necessarily Hausdorff. Now we introduce a particular form of Park [9, Theorem 1] which is modified into convenient form in order to derive main results.

THEOREM A. *Let X be a (not necessarily Hausdorff) convex space, $G : X \to 2^X$ a multifunction, and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, Gx is convex;*

(2) *for each $x \in K$, Gx is nonempty;*

(3) *for each $y \in X$, $G^{-1}y$ is compactly open;*

(4) *for each $N \in \mathscr{F}(X)$, there exists an $L_N \in kc(X)$ containing N such that for each $x \in L_N \setminus K$, $Gx \cap L_N \neq \emptyset$. Then G has a fixed point x_0; that is, $x_0 \in Gx_0$.*

3. MAIN RESULTS

Throughout this paper, every vector space is assumed to be real. We directly go to our main theorem.

THEOREM 1. *Let E be a topological vector space, X a nonempty convex subset of E, F be a vector space over R and $\langle\, ,\rangle : F \times E \to R$ be a bilinear functional. Suppose that*

(1.1) *$h : X \to R$ is a (not necessarily l.s.c.) convex function;*

(1.2) *$T : X \to 2^F$ is a nonempty-valued multifunction and for each $u \in X$, the set*

$$\left\{ x \in X \mid \inf_{z \in Tx} \langle z, x - u \rangle + h(x) - h(u) > 0 \right\}$$

is compactly open;

(1.3) *for each* $N \in \mathscr{F}(X)$, *there exists* $L_N \in kc(X)$ *containing* N *such that* $x \in L_N \setminus K$ *implies that there is a point* $u \in L_N$ *satisfying* $\inf_{z \in Tx}\langle z, x - u\rangle + h(x) - h(u) > 0$.

Then there exists an $x_o \in K$ *such that*

$$\inf_{z \in Tx_0} \langle z, x_0 - u\rangle \leq h(u) - h(x_0) \qquad \textit{for all } u \in X.$$

If, in addition, F *is equipped with the* $\sigma\langle F, E\rangle$*-topology, and* Tx_0 *is compact convex, then there exists* $z_0 \in Tx_0$ *such that*

$$\langle z_0, x_0 - u\rangle \leq h(u) - h(x_0) \qquad \textit{for all } u \in X.$$

Proof. Define a multifunction $G : X \to 2^X$ by

$$Gx = \left\{u \in X \mid \inf_{z \in Tx} \langle z, x - u\rangle + h(x) - h(u) > 0\right\}$$

for every $x \in X$. Then it is easy to show that Gx is convex for each $x \in X$. Also for each $u \in X$,

$$G^{-1}u = \left\{x \in X \mid \inf_{z \in Tx} \langle z, x - u\rangle + h(x) - h(u) > 0\right\}$$

is compactly open by (1.2). Moreover, G satisfies the condition (4) of Theorem A from (1.3). Hence all the requirements of Theorem A except (2) are satisfied. Note that G is a fixed point free multifunction. Therefore there must be an $x_0 \in K$ such that Gx_0 is empty, that is

$$\inf_{z \in Tx_0} \langle z, x_0 - u\rangle \leq h(u) - h(x_0) \qquad \text{for all } u \in X. \qquad (*)$$

On the other hand, define $S : X \times Tx_0 \to R$ by

$$S(u, z) = \langle z, x_0 - u\rangle + h(x_0) - h(u).$$

Then for each $u \in X$, $z \mapsto S(u, z)$ is continuous (the $\sigma\langle F, E\rangle$-topology is nothing else but the topology of pointwise convergence in E) and affine, and for each $z \in Tx_0$, $u \mapsto S(u, z)$ is concave on X. Thus by Kneser's minimax theorem [7], we have

$$\min_{z \in Tx_0} \sup_{u \in X} \left[\langle z, x_0 - u\rangle + h(x_0) - h(u)\right]$$
$$= \sup_{u \in X} \min_{z \in Tx_0} \left[\langle z, x_0 - u\rangle + h(x_0) - h(u)\right] \leq 0$$

by $(*)$. Since Tx_0 is compact, there exists a $z_0 \in Tx_0$ such that

$$\sup_{u \in X} \left[\langle z, x_0 - u\rangle + h(x_0) - h(u)\right]$$
$$= \min_{z \in Tx_0} \sup_{u \in X} \left[\langle z, x_0 - u\rangle + h(x_0) - h(u)\right] \leq 0.$$

Therefore $\langle z_0, x_0 - u\rangle \le h(u) - h(x_0)$ for all $u \in X$. This completes our proof.

If we strengthen the topology $\sigma\langle F, E\rangle$ into $\delta\langle F, E\rangle$ and impose some continuity conditions on T and h, we get the following.

THEOREM 2 (cf. Ding and Tan [4, Theorem 4]). *Let E be a topological vector space, X a nonempty convex subset of E, F be a vector space and $\langle\, ,\rangle : F \times E \to R$ be a bilinear functional such that for each $z \in F$, $x \mapsto \langle z, x\rangle$ is continuous on X, Equip F with the $\delta\langle F, E\rangle$-topology. Suppose that*

(2.1) *$h : X \to R$ is a l.s.c. convex function;*

(2.2) *$T : X \to 2^F$ is u.s.c. such that for each $x \in X$, Tx is nonempty compact convex;*

(2.3) *for each $N \in \mathscr{F}(X)$, there exists $L_N \in kc(X)$ containing N such that $x \in L_N \setminus K$ implies that there is a point $u \in L_N$ satisfying $\inf_{z\in Tx}\langle z, x - u\rangle + h(x) - h(u) > 0$.*

Then there exists $z_0 \in Tx_0$ such that

$$\langle z_0, x_0 - u\rangle \le h(u) - h(x_0) \qquad \text{for all } u \in X.$$

Proof. Define a multifunction $G : X \to 2^X$ by

$$Gx = \left\{u \in X \mid \inf_{z\in Tx} \langle z, x - u\rangle + h(x) - h(u) > 0\right\}$$

as in Theorem 1. It suffices to show that for each $u \in X$, $G^{-1}u$ is compactly open in X. Indeed, for each fixed $u \in X$, each compact subset A of X, the function $f : A \to R$ defined by

$$fx := \inf_{z\in Tx} \langle z, x - u\rangle = -\sup_{z\in Tx} \langle z, u - x\rangle$$

is l.s.c. on A by means of Aubin and Cellina [1, Theorem 5, p. 52] and Kum [8, Lemma B]. Hence the function $g : A \to R$ defined by

$$gx := \inf_{z\in Tx} \langle z, x - u\rangle + h(x) - h(u)$$

is l.s.c. on A. Thus $G^{-1}u \cap A = \{x \in A \mid gx > 0\}$ is open in A, as desired. Repeating the same argument as in Theorem 1, we have the conclusion.

The following example shows that the continuity conditions (2.1) and (2.2) are not essential, hence can be completely removed. Thus Theorem 1 generalizes Theorem 2 strictly.

EXAMPLE. Define $T : [0, 1\} \to 2^R$ by

$$Tx = \begin{cases} [-3, -2] & \text{if } x = 0 \\ \{-1\} & \text{if } 0 < x \le 1, \end{cases}$$

and define $h:[0,1] \to T$ by

$$hx = \begin{cases} 0 & \text{if } x = 0 \\ -1 & \text{if } 0 < x \leq 1. \end{cases}$$

For each $u \in [0,1]$, we have

$$\inf_{z \in Tx} \langle z, x - u \rangle + h(x) - h(u) = \begin{cases} 2u - h(u) & \text{if } x = 0 \\ u - x - 1 - h(u) & \text{if } 0 < x \leq 1. \end{cases}$$

It can be easily checked that for each $u \in [0,1]$, the set

$$G^{-1}u = \left\{ x \in X \mid \inf_{z \in Tx} \langle z, x - u \rangle + h(x) - h(u) > 0 \right\}$$

is open in [0, 1]. Indeed, $G^{-1}u = \varnothing$ if $u = 0$, and $G^{-1}u = [0, u)$ if $0 < u \leq 1$. Therefore, by Theorem 1, there exists an $x_0 \in [0,1]$ and $z_0 \in Tx_0$ such that $\langle z_0, x_0 - u \rangle \leq h(u) - h(x_0)$ for all $u \in [0, 1\}$. In fact, $x_0 = 1$ and $z_0 = -1$ satisfies the above inequality. On the other hand, T is neither u.s.c. nor l.s.c. Moreover, h is not l.s.c. Hence Theorem 2 cannot apply in this case.

Remark. Theorem 2 slightly improves Ding and Tan [4, Theorem 4] in the sense that the coercive condition (2.3) is more general than (c) of Theorem 4 in Ding and Tan [4]. Indeed, taking $L_N = \text{co}(X_0 \cup N)$ for each $N \in \mathscr{F}(X)$, then we can easily show that the coercive condition (c) of Theorem 4 in Ding and Tan [4] implies (2.3). Consequently, our example shows that Theorem 1 generalizes Ding and Tan [4, Theorem 4] strictly.

4. CONSEQUENCES OF THEOREM 1

The following Euclidean version of Theorem 1 also extends Yao and Guo [10, Theorem 4.5] in the sense that the coercive condition is weakened. So we can say that Theorem 1 is an infinite dimensional generalization of Yao and Guo [10, Theorem 4.5].

THEOREM 3. *Let X be a nonempty convex subset of R^n, and K a nonempty compact subset of X. Let $T: X \to 2^{R^n}$ be a nonempty-valued multifunction. Suppose that for each $u \in X$, the set*

$$\left\{ x \in X \mid \inf_{z \in Tx} \langle z, x - u \rangle > 0 \right\}$$

is compactly open. Suppose further that for each $N \in \mathscr{F}(X)$, there exists $L_N \in kc(X)$ containing N such that $x \in L_N \setminus K$ implies that there is a point $u \in L_N$ satisfying $\inf_{z \in Tx} \langle z, x - u \rangle > 0$. Then there exists an $x_0 \in X$ such that

$$\inf_{z \in Tx_0} \langle z, x_0 - u \rangle \leq 0 \qquad \textit{for all } u \in X.$$

If, in addition, Tx_0 is compact convex, then there exists $z_0 \in Tx_0$ such that

$$\langle z_0, x_0 - u \rangle \leq 0 \qquad \text{for all } u \in X.$$

Here $\langle\ ,\ \rangle$ denotes the inner product on R^n.

Proof. Putting $E = F = R^n$, $\langle\ ,\ \rangle$ the inner product on R^n, and $hx \equiv 0$ in Theorem 1, we get the result immediately.

Remark. Let us compare two coercive conditions, the one in Theorem 3 and the other in Yao and Guo [10, Theorem 4.5] as follows;

(I) "For each $N \in \mathscr{F}(X)$, there exists $L_N \in kc(X)$ containing N such that $x \in L_N \setminus K$ implies that there is a point $u \in L_N$ satisfying $\inf_{z \in Tx}\langle z, x - u \rangle > 0$" (Theorem 3).

(II) "There exists a nonempty bounded subset D of X a closed convex subset of R^n such that for any $x \in X \setminus D$ there exists $u \in D$ with the property $\inf_{z \in Tx}\langle z, x - u \rangle > 0$" (Yao and Guo [10, Theorem 4.5]).

In fact, if we take the compact subset $K = \operatorname{cl} D$ the closure of D and $L_N = \overline{\operatorname{co}}(N \cup \operatorname{cl} D)$ the closed convex hull of $N \cup \operatorname{cl} D$ for each $N \in \mathscr{F}(X)$, then K and L_N are clearly subsets of the closed convex set X because R^n is complete, whence (II) implies (I). Thus we can conclude that Theorem 3 refines Yao and Guo [10, Theorem 4.5].

THEOREM 4. *Let E, F, X, T, and $\langle\ ,\ \rangle$ be the same as in Theorem 1 except that X is compact convex. Suppose that for each $u \in X$, the set*

$$\left\{x \in X \mid \inf_{z \in Tx} \langle z, x - u \rangle > 0\right\}$$

is open in X. Then there exists an $x_0 \in X$ such that

$$\inf_{z \in Tx_0} \langle z, x_0 - u \rangle \leq 0 \qquad \text{for all } u \in X.$$

If, in addition, F is equipped with the $\sigma\langle F, E\rangle$-topology and Tx_0 is compact convex, then there exists $z_0 \in Tx_0$ such that

$$\langle z_0, x_0 - u \rangle \leq 0 \qquad \text{for all } u \in X.$$

Proof. Putting $L_N = K = X$, and $hx \equiv 0$ in Theorem 1, we obtain the result.

Remark. In particular, if $E = F = R^n$ and $\langle\ ,\ \rangle$ is the inner product on R^n, Theorem 4 goes back to Yao and Guo [10, Theorem 4.1]. Therefore Theorem 4 is a purely infinite-dimensional version of Yao and Guo.

We conclude this section with a generalization of Browder [3, Theorem 6] without assuming the local convexity of the underlying t.v.s. E.

THEOREM 5. *Let E be a Hausdorff topological vector space and X a nonempty compact convex subset of E. Let E^* be the topological dual space of E equipped with the topology of uniform convergence on each compact subset $K \subset E$ (or the strong topology), and $\langle\, ,\, \rangle : E^* \times E \to R$ the usual pairing. Let $T : X \to kc(E^*)$ be an u.s.c. multifunction. Then there exists an $x_0 \in X$ and $z_0 \in Tx_0$ such that*

$$\langle z_0, x_0 - u \rangle \le 0 \qquad \text{for all } u \in X.$$

Proof. Put $F = E^*$, the topological dual space of E, and $\langle\, ,\, \rangle$ the usual pairing on $E^* \times E$. Just mimicking the proof of Theorem 2, we see that for each $u \in X$, the set $\{x \in X \mid \inf_{z \in Tx} \langle z, x - u \rangle > 0\}$ is open. All the requirements of Theorem 4 are satisfied. Therefore we have the result.

Remark. Browder, to obtain his result [3, Theorem 6], depended heavily upon an auxiliary lemma [3, Lemma 1] which is a little difficult to derive without invoking relatively sophisticated results. However, we only appealed to the well-known Kneser's minimax theorem in the proof of Theorems 1 and 2 instead.

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