

# GENERALIZED VARIATIONAL INEQUALITIES OF THE HARTMAN-STAMPACCHIA-BROWDER TYPE

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ABSTRACT. We obtain several generalized variational inequalities from an equilibrium theorem due to the first author under more weaker hypothesis and in more general setting than known ones. Our new results extend, unify and improve many known Hartman-Stampacchia-Browder type variational inequalities for u.s.c. or monotone type multimaps. Our proofs are also much simpler than known ones.

## 1. Introduction

The variational inequalities due to Hartman and Stampacchia [15] and Browder [4],[5] have been extended by many scholars and applied to many problems in mathematical sciences. For the literature, see [9].

In a recent work of the first author [22], an equilibrium theorem is obtained within the frame of the KKM theory. This result extends known variational inequalities due to Brézis-Nirenberg-Stampacchia, Juberg-Karamardian, Mosco, Allen, Takahashi, Gwinner, Lassonde, Park, and Ben-El-Mechaiekh. For the literature, see [22]. On the other hand, more recently, many authors obtained a lot of the Hartman-Stampacchia-Browder type variational inequalities for upper semicontinuous or monotone type multimaps and their applications.

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In the present paper, first of all, we deduce three generalized variational inequality theorems from the first author's equilibrium theorem, under weaker hypotheses and in more general setting than known ones. Our new results extend, unify and improve the Hartman-Stampacchia-Browder type variational inequalities due to Hartman-Stampacchia [15], Lions-Stampacchia [18], Stampacchia [29], Mosco [19], Browder [4],[5], Allen [1], Shih-Tan [26], Park [20],[21], Ding-Tan [12], Yao-Guo [33], Chang-Zhang [7],[8], and Park-Kum [24].

In the second part, by adopting a generalized concept of monotonicity, we give simplified and generalized versions of the monotone type variational inequalities with much simpler proofs. These new results generalize works of Tarafdar-Yuan [31], Chang *et al.* [6], Ding-Tan [12], Tan-Yuan [30], and others.

## 2. Preliminaries

An extended real-valued function  $g : X \rightarrow \overline{\mathbf{R}}$  defined on a topological space  $X$  is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if  $\{x \in X : gx > r\}$  [resp.  $\{x \in X : gx < r\}$ ] is open for each  $r \in \overline{\mathbf{R}}$ .

For topological spaces  $X$  and  $Y$ , a multimap  $T : X \multimap Y$  is a function from  $X$  into the set  $2^Y$  of nonempty subsets of  $Y$ . We say that  $T$  is *lower semicontinuous* (l.s.c.) at  $x_0 \in X$  [B] if for each open set  $G$  with  $Tx_0 \cap G \neq \emptyset$ , there exists a neighborhood  $U$  of  $x_0$  such that  $x \in U$  implies  $Tx \cap G \neq \emptyset$ ; and *upper semicontinuous* (u.s.c.) at  $x_0 \in X$  [B] if for each open set  $G$  with  $Tx_0 \subset G$ , there exists a neighborhood  $U$  of  $x_0$  such that  $x \in U$  implies  $Tx \subset G$ . We say that  $T$  is l.s.c. [u.s.c.] if it is l.s.c. [u.s.c.] at each point of  $X$ .

For a convex subset  $X$  of a topological vector space (simply, t.v.s.)  $E$ , let  $k(X)$  denote the set of all nonempty compact subsets of  $X$ , and  $kc(X)$  all nonempty compact convex subsets.

The following is well-known. See Berge [3].

**Lemma 1.** *Let  $X$  and  $Y$  be topological spaces,  $g : X \times Y \rightarrow \mathbf{R}$  l.s.c., and  $T : X \rightarrow k(Y)$  u.s.c. Then the function  $U : X \rightarrow [-\infty, \infty)$  defined by*

$$U(x) = \inf_{y \in T_x} g(x, y)$$

*is l.s.c.*

Let  $\mathbf{K}$  denote either the real or complex field.

From Lemma 1 we have the following immediately:

**Lemma 2.** *Let  $E$  be a t.v.s. over  $\mathbf{K}$ ,  $X$  a nonempty subset of  $E$ ,  $F$  a topological space,  $T : X \rightarrow k(F)$  u.s.c., and  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbf{K}$  a function such that, for each  $y \in E$ ,  $(z, x) \mapsto \operatorname{Re}\langle z, x - y \rangle$  is l.s.c. on  $F \times X$ . Then for each  $y \in E$ , the function  $U : X \rightarrow [-\infty, \infty)$  defined by*

$$U(x) = \inf_{z \in T_x} \operatorname{Re}\langle z, x - y \rangle$$

*is l.s.c. on  $X$ .*

**Remark.** Lemma 2 contains Shih and Tan [28, Lemma 2], Ding and Tan [12, Lemma 1], Kim and Tan [16, Lemmas 2 and 4], and Chang and Zhang [8, Lemma 3] as particular cases.

Let  $E$  and  $F$  be vector spaces over  $\mathbf{K}$  and  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbf{K}$  a bilinear functional. For each  $x \in E$ , each nonempty subset  $A$  of  $E$ , and each  $\varepsilon > 0$ , let

$$W(x, \varepsilon) = \{z \in F : |\langle z, x \rangle| < \varepsilon\},$$

$$U(A, \varepsilon) = \{z \in F : \sup_{x \in A} |\langle z, x \rangle| < \varepsilon\}.$$

Let  $\sigma(F, E)$  denote the topology on  $F$  generated by the family  $\{W(x, \varepsilon) : x \in E, \varepsilon > 0\}$  as a subbase for the neighborhood system at 0. Similarly, we can define

the topology  $\sigma(E, F)$  on  $E$ . If  $E$  is a t.v.s., let  $\delta(F, E)$  denote the topology on  $F$  generated by the family

$$\{U(B, \varepsilon) : B \text{ is a nonempty compact subset of } E \text{ and } \varepsilon > 0\}$$

as a base for the neighborhood system at 0. If  $E$  is a t.v.s., let  $\eta(F, E)$  denote the topology on  $F$  generated by the family

$$\{U(B, \varepsilon) : B \text{ is a nonempty bounded subset of } E \text{ and } \varepsilon > 0\}$$

as a base for the neighborhood system at 0.

If  $F$  possesses the topology  $\sigma(F, E)$  or  $\delta(F, E)$ , then  $F$  becomes a locally convex t.v.s., not necessarily Hausdorff. If  $F$  possesses the topology  $\eta(F, E)$ ,  $F$  becomes a t.v.s.

**Lemma 3.** *Let  $E$  be a t.v.s. over  $\mathbf{K}$ ,  $F$  a vector space over  $\mathbf{K}$ ,  $C$  a nonempty subset of  $E$ , and  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbf{K}$  a bilinear functional. Suppose that for each  $z \in F$ ,  $y \mapsto \langle z, y \rangle$  is continuous on  $C$  and that one of the following holds:*

- (A)  $F$  has  $\sigma(F, E)$ -topology
- (B)  $C$  is compact and  $F$  has  $\delta(F, E)$ -topology.
- (C)  $C$  is bounded and  $F$  has  $\eta(F, E)$ -topology.

*If  $T : C \rightarrow k(F)$  is u.s.c., then for each  $y \in E$  the function*

$$x \mapsto \inf_{z \in Tx} \operatorname{Re} \langle z, x - y \rangle$$

*is l.s.c. on  $C$ .*

*Proof.* (A) For each  $y \in E$ ,  $(z, x) \mapsto \operatorname{Re} \langle z, x - y \rangle$  is continuous on  $F \times C$ . Therefore, by Lemma 2, we have the conclusion.

(B), (C) As in the proof of Kum [17, Lemma B], the pairing  $\langle \cdot, \cdot \rangle : F \times C \rightarrow \mathbf{K}$  is continuous. Therefore, by Lemma 2, we have the conclusion.

**Remark.** If  $F = E^*$ , the topological dual of  $E$ , then  $y \mapsto \langle z, y \rangle$  is continuous automatically.

**Particular Forms.** 1. Browder [4, Lemma 1]:  $E$  is locally convex,  $C$  is compact,  $F = E^*$ ,  $\langle \cdot, \cdot \rangle$  is the pairing between  $E^*$  and  $E$ , and  $T$  is single-valued.

2. Shih and Tan [26, Lemma 1]: Lemma 3(C) for a locally convex t.v.s.  $E$ ,  $F = E^*$ , and a single-valued continuous map  $T$ .

3. Kim and Tan [16, Lemma 2]: Lemma 3(C) with  $F = E^*$ .

4. Chang and Zhang [8, Lemma 3] and Zhang [34, Theorem 1] obtained Lemma 3(C) with a proof more lengthy than ours.

The following is well-known:

**Lemma 4.** (Ky Fan [14]) *Let  $X$  be a compact Hausdorff space and  $Y$  a set. Let  $f$  be a real-valued function on  $X \times Y$  such that for every  $y \in Y$ ,  $f(x, y)$  is l.s.c. on  $X$ . If  $f$  is convex on  $X$  and concave on  $Y$ , then*

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$

For the terminology in Lemma 4, see [14].

A *convex space*  $X$  is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets.

A subset  $B$  of a topological space  $X$  is said to be *compactly closed* in  $X$  if for every compact set  $K \subset X$  the set  $B \cap K$  is closed in  $K$ .

Let  $\langle X \rangle$  denote the set of all nonempty finite subsets of  $X$ , and  $\text{co}$  and  $\overline{\quad}$  denote the convex hull and closure, resp.

The following equilibrium theorem is the basis of our arguments:

**Theorem 0.** *Let  $X$  be a convex space,  $f, g : X \times Y \rightarrow [-\infty, \infty]$ ,  $K$  a nonempty compact subset of  $X$ , and  $\gamma \in [-\infty, \infty]$ . Suppose that*

- (0.1)  $f(x, x) \leq \gamma$  for all  $x \in X$ ;
- (0.2) for each  $y \in X$ ,  $\{x \in X : g(x, y) > \gamma\}$  is compactly open;
- (0.3) for each  $x \in X$ ,  $\{y \in X : f(x, y) > \gamma\} \supset \text{co}\{y \in X : g(x, y) > \gamma\}$ ; and
- (0.4) for each  $N \in \langle D \rangle$ , there exists a compact convex subset  $L_N$  of  $X$  containing  $N$  such that for each  $x \in L_N \setminus K$ ,  $g(x, y) > \gamma$  for some  $y \in L_N$ .

*Then there exists an  $\bar{x} \in K$  such that  $g(\bar{x}, y) \leq \gamma$  for all  $y \in X$ .*

Note that Theorem 0 follows from [22, Theorem 9] and is slightly different from [22, Theorem 10]. A far-reaching generalization of Theorem 0 is given in [23, Theorem 6].

### 3. Variational inequalities for u.s.c. multimaps

From Theorem 0, we deduce a number of known results on variational inequalities.

**Theorem 1.** *Let  $X$  be a convex space,  $E$  a vector space over  $\mathbf{K}$  containing  $X$  as a subset,  $Z$  a set,  $T : X \multimap Z$  a multimap, and  $K$  a nonempty compact subset of  $X$ . Suppose that*

- (1.1)  $\langle \cdot, \cdot \rangle : Z \times E \rightarrow \mathbf{K}$  is a function such that, for each  $z \in Z$ ,  $\langle z, \cdot \rangle$  is linear on  $X$ ;
- (1.2)  $\alpha : X \times X \rightarrow \mathbf{R}$  is a function such that, for each  $x \in X$ ,  $\alpha(x, x) = 0$  and  $\alpha(x, \cdot)$  is concave;
- (1.3) for each  $y \in X$ , the set

$$\{x \in X : \inf_{z \in T_x} \text{Re}\langle z, x - y \rangle + \alpha(x, y) > 0\}$$

*is compactly open; and*

- (1.4) for each  $N \in \langle X \rangle$ , there exists an  $L_N \in kc(X)$  containing  $N$  such that  $x \in L_N \setminus K$  implies

$$\inf_{z \in T_x} \text{Re}\langle z, x - y \rangle + \alpha(x, y) > 0 \quad \text{for some } y \in L_N.$$

Then there exists an  $\bar{x} \in K$  such that

$$\inf_{z \in T\bar{x}} \operatorname{Re}\langle z, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

Moreover, the set of all solutions  $\bar{x}$  is a compact subset of  $K$ .

*Proof.* Define  $p : X \times X \rightarrow [-\infty, \infty]$  by

$$p(x, y) = \inf_{z \in Tx} \operatorname{Re}\langle z, x - y \rangle + \alpha(x, y) \quad \text{for } (x, y) \in X \times X.$$

We use Theorem 0 with  $p = f = g$  and  $\gamma = 0$ .

- (1)  $p(x, x) = 0$  for  $x \in X$  by (1.1) and (1.2).
- (2) For each  $y \in X$ ,  $\{x \in X : p(x, y) > 0\}$  is compactly open by (1.3).
- (3) For each  $x \in X$ ,

$$\{y \in X : p(x, y) > 0\} = \{y \in X : \inf_{z \in Tx} \operatorname{Re}\langle z, x - y \rangle + \alpha(x, y) > 0\}$$

is convex since  $\langle z, \cdot \rangle$  is linear and  $\alpha(x, \cdot)$  is concave by (1.1) and (1.2).

- (4) (1.4)  $\implies$  (0.4).

Therefore, by Theorem 0, we have the first conclusion. Moreover, the set of all solutions  $\bar{x}$  is in the intersection  $\bigcap_{y \in X} \{x \in K : p(x, y) \leq 0\}$  of the compactly closed subsets of  $K$ , and hence compact. This completes our proof.

**Particular Forms.** 1. Allen [1, Corollary 1]:  $E$  is a t.v.s.,  $Z = E^*$ ,  $T : X \rightarrow E^*$  a function such that  $x \mapsto \langle Tx, x \rangle$  is l.s.c. on  $X$ , and  $\alpha(x, y) = f(x) - f(y)$ , where  $f : E \rightarrow (-\infty, \infty]$  is a l.s.c. convex function which is finite on  $X$ . Note that our coercivity condition is more general than Allen's.

2. Yao and Guo [33, Theorems 3.1 and 4.1]:  $E = Z = \mathbf{R}^n$  and  $\alpha = 0$ . Consequently, all of the existence results of variational problems in [33, Sections 3 and 4] are consequences of Theorem 1.

3. Park and Kum [23, Theorem 2]:  $Z$  is a vector space,  $\langle \cdot, \cdot \rangle : Z \times E \rightarrow \mathbf{R}$  is a bilinear functional, and  $\alpha = 0$ .

From Theorem 1 and Lemma 2, we obtain the following:

**Theorem 2.** Let  $X$  be a convex subset of a t.v.s.  $E$  over  $\mathbf{K}$ ,  $F$  a t.v.s. over  $\mathbf{K}$ ,  $T : X \rightarrow k(F)$  u.s.c., and  $K$  a nonempty compact subset of  $X$ . Suppose that

(2.1)  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$  is a function such that, for each  $z \in F$ ,  $\langle z, \cdot \rangle$  is linear on  $E$ ;

(2.2)  $\alpha : X \times X \rightarrow \mathbf{R}$  is a function such that, for each  $x \in X$ ,  $\alpha(x, x) = 0$ ,  $\alpha(x, \cdot)$  is concave, and  $\alpha(\cdot, x)$  is l.s.c. on compact subsets of  $X$ ;

(2.3) for each  $y \in X$ ,  $(z, x) \mapsto \operatorname{Re}\langle z, x - y \rangle$  is l.s.c. on  $F \times X$ ; and

(2.4) for each  $N \in \langle X \rangle$ , there exists an  $L_N \in kc(X)$  containing  $N$  such that  $x \in L_N \setminus K$  implies

$$\inf_{z \in Tx} \operatorname{Re}\langle z, x - y \rangle + \alpha(x, y) > 0 \quad \text{for some } y \in L_N.$$

Then there exists an  $\bar{x} \in K$  such that

$$\inf_{z \in T\bar{x}} \operatorname{Re}\langle z, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

Moreover, the set of all solutions  $\bar{x}$  is a compact subset of  $K$ .

*Proof.* We use Theorem 1 with  $Z = F$ . Note that (2.1)  $\iff$  (1.1), (2.4)  $\iff$  (1.4), and (2.2)  $\implies$  (1.2). Since  $T : X \rightarrow k(F)$  is u.s.c., by Lemma 2, (2.3) implies that, for each  $y \in X$ ,

$$x \mapsto \inf_{z \in Tx} \operatorname{Re}\langle z, x - y \rangle$$

is l.s.c. on compact subsets of  $X$ ; and hence

$$x \mapsto \inf_{z \in Tx} \operatorname{Re}\langle z, x - y \rangle + \alpha(x, y)$$

is l.s.c. on compact subsets of  $X$ . Therefore, (1.3) is satisfied. Now the conclusion follows from Theorem 1.

**Particular Forms.** 1. Hartman and Stampacchia [15, Lemma 3.1]:  $X = K$  is a compact convex subset of  $E = F = \mathbf{R}^n$ ,  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbf{R}^n$ ,  $T : X \rightarrow \mathbf{R}^n$  a continuous map, and  $\alpha = 0$ .

2. Lions and Stampacchia [18], Stampacchia [29], and Mosco [19]:  $X = K$  is a compact convex subset of a real inner product space  $E$ ,  $T = 1_X$ ,  $a : E \times E \rightarrow \mathbf{R}$  a continuous bilinear form on  $E$ , and for a  $v' \in E^*$ , let  $\alpha(u, w) = \langle v', w - u \rangle$ . Then there exists a  $u \in K$  such that

$$a(u, w - u) \leq \langle v', u - w \rangle \quad \text{for all } w \in X.$$

3. Park [21, Corollary 2.1]:  $T$  is single-valued.

From Theorem 1 and Lemmas 3 and 4, we obtain the following:



**Theorem 3.** Let  $X$  be a convex subset of a t.v.s.  $E$  over  $\mathbf{K}$ ,  $K$  a nonempty compact subset of  $X$ , and  $F$  a vector space over  $\mathbf{K}$ . Suppose that

(3.1)  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi$  is a bilinear functional such that, for each  $z \in F$ ,  $x \mapsto \langle z, x \rangle$  is continuous on each compact subset of  $X$ ;

(3.2)  $\alpha : X \times X \rightarrow \mathbf{R}$  is a function as in (2.2);

(3.3)  $F$  has any one of topologies  $\sigma(F, E)$ ,  $\delta(F, E)$  and  $\eta(F, E)$ ; and  $T : X \rightarrow k(F)$  is u.s.c. on compact subsets of  $X$ ; and

(3.4) for each  $N \in \langle X \rangle$ , there exists an  $L_N \in kc(X)$  containing  $N$  such that  $x \in L_N \setminus K$  implies

$$\inf_{z \in Tx} \operatorname{Re} \langle z, x - y \rangle + \alpha(x, y) > 0 \quad \text{for some } y \in L_N.$$

Then there exists an  $\bar{x} \in K$  such that

$$\inf_{z \in T\bar{x}} \operatorname{Re} \langle z, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

Moreover, the set of solutions  $\bar{x}$  is a compact subset of  $K$ . Further, if  $T\bar{x}$  is Hausdorff and convex, then there exists a  $\bar{z} \in T\bar{x}$  such that

$$\operatorname{Re} \langle \bar{z}, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

*Proof.* We use Theorem 1 with  $F = Z$ . Then (3.1)  $\implies$  (1.1), (3.2)  $\implies$  (1.2), and (3.4)  $\iff$  (1.4). It remains to show that (1.3) holds. For any compact subset  $C$  of  $X$ , by Lemma 3, (3.1) and (3.3) imply that, for each  $y \in X$ , the function

$$x \mapsto \inf_{z \in Tx} \operatorname{Re} \langle z, x - y \rangle$$

is l.s.c. on  $C$ . Since  $\alpha(\cdot, y)$  is l.s.c. on  $C$  by (3.2),

$$x \mapsto \inf_{z \in Tx} \operatorname{Re} \langle z, x - y \rangle + \alpha(x, y)$$

is l.s.c. on  $C$ . Hence (1.3) is satisfied. Therefore, by Theorem 1, the first and second part of the conclusion holds.

Suppose that  $T\bar{x}$  is convex. Define  $f : T\bar{x} \times X \rightarrow \mathbf{R}$  by

$$f(z, y) = \operatorname{Re}\langle z, \bar{x} - y \rangle + \alpha(\bar{x}, y) \quad \text{for } (z, y) \in T\bar{x} \times X.$$

Then  $f$  is linear in  $z \in T\bar{x}$  and in  $y \in X$ . Moreover, for each  $y \in X$ ,  $z \mapsto \langle z, \bar{x} - y \rangle$  is continuous as in the proof of Lemma 3, and hence  $z \mapsto f(z, y)$  is l.s.c. on  $T\bar{x}$ . Therefore, by Lemma 4, we have

$$\min_{z \in T\bar{x}} \sup_{y \in X} f(z, y) = \sup_{y \in X} \min_{z \in T\bar{x}} f(z, y).$$

Since  $z \mapsto \sup_{y \in X} f(z, y)$  is l.s.c. on the compact set  $T\bar{x}$ , being the supremum of l.s.c. functions, there exists a  $\bar{z} \in T\bar{x}$  such that

$$\sup_{y \in X} f(\bar{z}, y) = \min_{z \in T\bar{x}} \sup_{y \in X} f(z, y) = \sup_{y \in X} \min_{z \in T\bar{x}} f(z, y).$$

Since the right hand side is  $\leq 0$  by the first part of the conclusion, we have  $\sup_{y \in X} f(\bar{z}, y) \leq 0$ ; that is,

$$\operatorname{Re}\langle \bar{z}, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

This completes our proof.

**Remark.** If  $F = E^*$ , the topological dual of  $E$ , then we do not need to assume the continuity of  $y \mapsto \langle z, y \rangle$  in (3.1).

**Particular Forms.** 1. Browder [4, Theorem 3], [5, Theorem 2]:  $E$  is locally convex,  $F = E^*$ ,  $X = K$ ,  $T : K \rightarrow E^*$  continuous, and  $\alpha = 0$ .

2. Browder [5, Theorem 6]:  $E$  is locally convex,  $F = E^*$ ,  $X = K$ ,  $T : K \rightarrow kc(E^*)$  u.s.c., and  $\alpha = 0$ .

3. Shih and Tan [26, Theorem 10]:  $E$  is locally convex,  $F = E^*$ ,  $T : X \rightarrow E^*$  continuous, and  $\alpha \equiv 0$ .

4. Park [20, Theorem 2]:  $X = K$  is a compact convex subset of a real t.v.s.  $E$ ,  $F = E^*$ ,  $T : K \rightarrow E^*$  continuous, and  $\alpha = 0$ .

5. Ding and Tan [12, Theorem 4]: Equip  $F$  with the  $\delta(F, E)$ -topology,  $\alpha(x, y) = h(x) - h(y)$  where  $h : X \rightarrow \mathbf{R}$  is a l.s.c. convex function, and assumed more restrictive coercivity condition.

6. Ding [11, Theorem 2.2]:  $X$  is equipped with the  $\sigma(E, F)$ -topology,  $F$  with the  $\sigma(F, E)$ -topology, and assumes stronger coercivity.

7. Park [21, Theorem 3]:  $E$  is locally convex,  $F = E^*$ ,  $T$  is single-valued, and  $\alpha = 0$ .

8. Chang and Zhang [8, Corollary 1]:  $X = K$  and  $\alpha(x, y) = h(x) - h(y)$ , where  $h : X \rightarrow \mathbf{R}$  is a l.s.c. convex function.

9. Zhang [34, Theorem 6]:  $F$  has the  $\delta(F, E)$ -topology,  $\alpha(x, y) = h(x) - h(y)$  as above, and the coercivity is stronger than ours.

10. Park and Kum [24, Theorem 1]:  $F = E^*$  with  $\delta(F, E)$ -topology,  $X = K$ , and  $\alpha = 0$ .

#### 4. Variational inequalities for generalized monotone multimaps

Let  $E$  and  $F$  are t.v.s. and  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbf{K}$  be a pairing. For any  $X \subset E$ , a multimap  $T : X \multimap F$  is said to be

(i) *monotone* if for each  $x, y \in X$ ,  $u \in Tx$ , and  $v \in Ty$ ,

$$\operatorname{Re}\langle u - v, x - y \rangle \geq 0;$$

(ii) *semimonotone* [2] if for each  $x, y \in X$ ,  $u \in Tx$ , and  $v \in Ty$ ,

$$\inf_{u \in Tx} \operatorname{Re}\langle u, x - y \rangle \geq \inf_{v \in Ty} \operatorname{Re}\langle v, x - y \rangle;$$

(iii) *quasimonotone* [12] if for each  $x, y \in X$ ,

$$\inf_{v \in Ty} \operatorname{Re}\langle v, x - y \rangle > 0 \text{ implies } \inf_{u \in Tx} \operatorname{Re}\langle u, x - y \rangle > 0.$$

The concepts of semimonotonicity or quasimonotonicity are used in the works of Bae *et al.* [2], Cottle and Yao [10], Yao [32], and Ding-Tarafdar [13].

We now introduce the following:  $T$  is said to be

(iv) *demimonotone* if for each  $x, y \in X$ ,

$$\sup_{v \in Ty} \operatorname{Re}\langle v, x - y \rangle > 0 \text{ implies } \inf_{u \in Tx} \operatorname{Re}\langle u, x - y \rangle > 0.$$

Note that this kind of monotonicity is used by Zhang [34, Theorems 4,7, and 8] and extends the monotonicity (i).

In this section, from Theorem 0, we deduce generalized and simplified versions of the main results of [6] and many other authors with much simpler proofs.

The following existence result on solutions of variational inequalities for demimonotone multimaps is important:

**Theorem 4.** *Let  $X$  be a convex subset of a t.v.s.  $E$  over  $\mathbf{K}$ ,  $K$  a nonempty compact subset of  $X$ ,  $F$  a vector space over  $\mathbf{K}$ , and  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbf{K}$  a bilinear functional such that for each  $f \in F$ , the function  $x \mapsto \langle f, x \rangle$  is l.s.c. on  $X$ .*

*Suppose that*

(4.1)  $T : X \multimap F$  is demimonotone;

(4.2)  $\alpha : X \times X \rightarrow \mathbf{R}$  is such that for each  $y \in X$ ,  $\alpha(\cdot, y)$  is l.s.c. on compact subsets of  $X$ ;

(4.3) for each  $N \in \langle X \rangle$ , there exists a compact convex subset  $L_N$  of  $X$  containing  $N$  such that for each  $x \in L_N \setminus K$ ,

$$\sup_{v \in Ty} \operatorname{Re}\langle v, x - y \rangle + \alpha(x, y) > 0 \quad \text{for some } y \in L_N.$$

*Then there exists an  $\bar{x} \in K$  such that*

$$\sup_{v \in Ty} \operatorname{Re}\langle u, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

*Proof.* For  $x, y \in X$ , let

$$\begin{aligned} g(x, y) &:= \sup_{v \in Ty} \operatorname{Re}\langle v, x - y \rangle + \alpha(x, y), \\ f(x, y) &:= \inf_{u \in Tx} \operatorname{Re}\langle u, x - y \rangle + \alpha(x, y), \end{aligned}$$

and use Theorem 0 with  $\gamma = 0$ . Note that (0.1) holds by (4.2).

Since  $T$  is demimonotone, we have  $g(x, y) > 0$  implies  $f(x, y) > 0$  for each  $x, y \in X$ . Note that  $\{y \in X : f(x, y) > 0\}$  is convex. In fact, if  $f(x, y_1) > 0$  and  $f(x, y_2) > 0$ , by the bilinearity of  $\langle \cdot, \cdot \rangle$  and (4.2), we can easily check that

$$f(x, ty_1 + (1 - t)y_2) \geq tf(x, y_1) + (1 - t)f(x, y_2) > 0$$

for  $0 < t < 1$ . Hence, we have

$$\{y \in X : f(x, y) > 0\} \supset \operatorname{co}\{y \in X : g(x, y) > 0\}$$

for each  $x \in X$ . This shows (0.3).

Note that for each  $z \in F$ ,  $x \mapsto \langle z, x \rangle$  is l.s.c. on  $X$  by assumption, and hence  $x \mapsto \sup_{v \in Ty} \langle v, x - y \rangle$  is l.s.c. for each  $y \in X$ , being the supremum of l.s.c. functions. Since  $\alpha(\cdot, y)$  is l.s.c. on compact subsets of  $X$  by (4.2),  $x \mapsto g(x, y)$  is l.s.c. on compact subsets of  $X$  for each  $y \in X$ . Hence (0.2) holds.

Moreover, (4.3) implies (0.4). Therefore, by Theorem 0, there exists an  $\bar{x} \in K$  such that  $\sup_{y \in X} g(\bar{x}, y) \leq 0$ ; that is,

$$\sup_{v \in Ty} \operatorname{Re}\langle v, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

This completes our proof.

**Remark.** For  $F = E^*$ , a monotone map  $T$ , and  $\alpha(x, y) = h(x) - h(y)$ , where  $h : X \rightarrow \mathbf{R}$  is a convex l.s.c. function, Theorem 4 reduces to Tarafdar and Yuan [31, Theorem 3.1], where the coercivity is stronger than ours.

**Lemma 5.** *Let  $E$  be a t.v.s. over  $\mathbb{K}$ ,  $F$  a vector space over  $\mathbb{K}$ , and  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbb{K}$  a bilinear functional. Then for each  $z \in F$ ,  $y \mapsto \langle z, y \rangle$  is continuous on  $E$  with the  $\sigma(E, F)$ -topology.*

For the proof, see [6, p.499].

From Theorem 4 and Lemma 5, we immediately obtain the following:

**Theorem 4'.** *Let  $X$  be a convex subset of a t.v.s.  $E$  over  $\mathbf{K}$ ,  $F$  a vector space over  $\mathbf{K}$  with the  $\sigma(F, E)$ -topology, and  $K$  a nonempty  $\sigma(E, F)$ -compact subset of  $X$  w.r.t. a bilinear functional  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbf{R}$ . Suppose that*

(4.1)'  $T : X \rightarrow F$  is demimonotone;

(4.2)'  $\alpha : X \times X \rightarrow \mathbf{R}$  is such that for each  $x \in X$ ,  $\alpha(x, x) = 0$  and  $\alpha(x, \cdot)$  is concave and for each  $y \in X$ ,  $\alpha(\cdot, y)$  is l.s.c. on compact subsets of  $X$  with the  $\sigma(E, F)$ -topology; and

(4.3)' for each  $N \in \langle X \rangle$ , there exists a  $\sigma(E, F)$ -compact convex subset  $L_N$  of  $X$  containing  $N$  such that for each  $x \in L_N \setminus K$ ,

$$\sup_{v \in T_y} \operatorname{Re} \langle v, x - y \rangle + \alpha(x, y) > 0 \quad \text{for some } y \in L_N.$$

Then there exists an  $\bar{x} \in K$  such that

$$\sup_{v \in T_y} \operatorname{Re} \langle v, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

**Lemma 6.** *Let  $X$  be a convex subset of a t.v.s.  $E$  over  $\mathbf{K}$ , and  $F$  a vector space over  $\mathbf{K}$  with the  $\sigma(F, E)$ -topology w.r.t. a bilinear functional  $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbf{K}$ . Suppose that*

(1)  $T : X \rightarrow F$  is u.s.c. on each line segment of  $X$ ; and

(2)  $\alpha : X \times X \rightarrow \mathbf{R}$  is a real function such that for each  $x \in X$ ,  $\alpha(x, x) = 0$  and  $\alpha(x, \cdot)$  is concave.

Then for each  $\bar{x} \in X$ , it follows from

$$\sup_{v \in T_y} \operatorname{Re} \langle v, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X$$

that

$$\inf_{u \in T_{\bar{x}}} \operatorname{Re} \langle u, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

*Proof.* Just follow the proof of Chang *et al.* [6, Lemma 3].

**Particular Forms.** 1. Shih and Tan [27, Lemma 2]:  $E$  is a Banach space,  $F = E^*$ ,  $\alpha = 0$ , and  $Tx$  is weak\* compact for each  $x \in X$ .

2. Tan and Yuan [30, Lemmas 3 and 8], Tarafdar and Yuan [31, Lemma 2.7]:  $F = E^*$ ,  $\alpha(x, y) = h(x) - h(y)$  where  $h : X \rightarrow \mathbf{R}$  is a convex function, and  $Tx$  is  $\sigma(E^*, E)$ -compact for each  $x \in X$ .

3. Chang *et al.* [6, Lemma 3]:  $\alpha(x, y) = h(x) - h(y)$ , where  $h : X \rightarrow \mathbf{R}$  is a convex function.

**Lemma 6'.** Let  $X, E, F$ , and  $\alpha$  be the same as in Lemma 6. Suppose that

(1)'  $T : X \multimap F$  is l.s.c. on each line segment of  $X$ .

Then for each  $\bar{x} \in X$ , it follows from

$$\sup_{v \in T_y} \operatorname{Re} \langle v, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X$$

that

$$\sup_{u \in T_{\bar{x}}} \operatorname{Re} \langle u, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

*Proof.* Just follow the proof of Ding and Tan [12, Lemma 3].

**Particular Forms.** 1. Ding and Tan [12, Lemma 3]:  $\alpha(x, y) = h(x) - h(y)$ , where  $h : X \rightarrow \mathbf{R}$  is a convex function..

2. Tan and Yuan [30, Lemma 2]:  $E$  is a Banach space,  $F = E^*$ , and  $\alpha(x, y) = h(x) - h(y)$  as above.

**Theorem 5.** *Under the hypothesis of Theorem 4 or 4', assume that*

$$(5.1) \quad T \text{ is u.s.c. on each line segment of } X \text{ to the } \sigma(F, E)\text{-topology on } F.$$

*Then there exists an  $\bar{x} \in K$  such that*

$$\inf_{u \in T\bar{x}} \operatorname{Re}\langle u, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

*Furthermore, if  $T\bar{x}$  is Hausdorff compact convex, then there exists a  $\bar{u} \in T\bar{x}$  such that*

$$\operatorname{Re}\langle \bar{u}, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

*Proof.* From Theorem 4 or 4' and Lemma 6, the first conclusion follows. For the last part, we can follow the proof of Theorem 3.

**Particular Forms.** 1. Stampacchia [29, Theorems 2.3 and 2.4]:  $E$  is a reflexive Banach space,  $X = K$ ,  $F = E^*$ ,  $\alpha = 0$ , and  $T$  is single-valued and monotone.

2. Shih and Tan [27, Theorems 1 and 2]:  $E$  is a reflexive Banach space,  $F = E^*$ ,  $\alpha = 0$ , and  $T$  is monotone with stronger coercivity.

3. Chang *et al.* [6, Theorem 2]:  $E$  is locally convex,  $\alpha(x, y) = h(x) - h(y)$  for a function  $h : X \rightarrow \mathbf{R}$ ,  $T$  is monotone, and the coercivity is stronger than ours.

4. Tarafdar and Yuan [31, Theorem 3.4(II)]:  $F = E^*$ ,  $\alpha(x, y) = h(x) - h(y)$  as above,  $T$  is monotone with stronger coercivity.

**Theorem 5'.** *Under the hypothesis of Theorem 4 or 4', assume that*

$$(5.1)' \quad T \text{ is l.s.c. on each line segment of } X \text{ to the } \sigma(F, E)\text{-topology on } F.$$

*Then there exists an  $\bar{x} \in K$  such that*

$$\sup_{u \in T\bar{x}} \operatorname{Re}\langle u, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \quad \text{for all } y \in X.$$

*Proof.* From Theorem 4 or 4' and Lemma 6', the conclusion follows.



**Particular Forms.** 1. Shih and Tan [26, Theorems 6 and 7]:  $F = E^*$ ,  $\alpha = 0$ , and  $T$  is monotone with stronger coercivity.

2. Ding and Tan [12, Theorems 2 and 3]:  $\alpha(x, y) = h(x) - h(y)$  for a function  $h : X \rightarrow \mathbf{R}$  and assume stronger coercivity.

3. Ding [11, Theorem 2.1]: Under stronger coercivity and additional assumption.

4. Zhang [34, Theorems 4 and 5]:  $\alpha(x, y) = h(x) - h(y)$  under stronger coercivity.

5. Tarafdar and Yuan [31, Theorem 3.4(I)]:  $F = E^*$ ,  $\alpha(x, y) = h(x) - h(y)$ , and assume stronger coercivity.

**Remarks.** 1. There are a lot of particular forms of (reflexive) Banach space versions of Theorems 4, 4', 5, and 5'. See [7],[31],[33] and others. Those results can be improved by following our method.

2. There are a lot of applications of Theorems 4, 4', 5 and 5'. See [7],[15],[18],[19],[29],[31],[33] and others. Some of them also can be improved using our results.

Finally, from Theorem 4 or 4', we have the following:

**Theorem 6.** *If  $T$  is monotone in Theorem 5, then there exists an  $\bar{x} \in X$  such that*

$$\sup_{y \in X} [\sup_{v \in Ty} \operatorname{Re}\langle v, \bar{x} - y \rangle + \alpha(\bar{x}, y)] \leq \sup_{y \in X} [\inf_{u \in T\bar{x}} \operatorname{Re}\langle u, \bar{x} - y \rangle + \alpha(\bar{x}, y)] \leq 0.$$

*Further, if  $T\bar{x}$  is Hausdorff compact convex, then there exists a  $\bar{u} \in T\bar{x}$  such that*

$$\operatorname{Re}\langle \bar{u}, \bar{x} - y \rangle + \alpha(\bar{x}, y) \leq 0 \text{ for all } y \in X.$$

*Proof.* Since  $T$  is monotone, for all  $x, y \in X$ ,  $u \in Tx$  and  $v \in Ty$ , we have

$$\operatorname{Re}\langle v, x - y \rangle \leq \operatorname{Re}\langle u, x - y \rangle$$

and hence we have

$$g(x, y) \leq f(x, y)$$

in the proof of Theorem 4, which readily implies the conclusion by Theorem 5.

**Remark.** Under the hypothesis of Theorem 4, a very particular form of Theorem 6 is due to Chang *et al.* [6, Theorem 1] and Chang and Zhang [7, Theorem 4.1].

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