Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 31, No. 1/2, pp. 207-216, 1998

0362-546X/98 $19.00+0.00

PII: S0362-546X(96)00305-7

# GENERALIZED VARIATIONAL INEQUALITIES AND FIXED POINT THEOREMS†

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*(Received 18 April 1996; revised 15 September 1996; received for publication 27 November 1996)*

*Key words and phrases*: Variational inequality, fixed point, Hausdorff topological vector space (t.v.s.), multimap (map), closed map, compact map, convexly totally bounded (c.t.b.), locally convex subset, $Z$ type, inward set, generalized upper hemicontinuous (u.h.c.) map, generalized nonlinear variational inequality problem (GNVIP).

## 1. INTRODUCTION

In this paper, from a well-known fixed point theorem of Idzik [1], we deduce new variational inequality for not-necessarily locally convex topological vector spaces and fixed point theorems for generalized upper hemicontinuous multimaps. Consequently, all of the results in Kim and Tan [2], Ding [3], and some others are substantially extended and improved. Moreover, our new fixed point theorems on generalized upper hemicontinuous multimaps include a large number of historically well-known extensions of the Brouwer or Kakutani theorems.

Throughout this paper, a t.v.s is a Hausdorff topological vector space, and co denotes the convex hull.

A *multimap* or *map* $T: X \multimap Y$ is a function from $X$ into the power set of $Y$ with nonempty values, and $x \in T^-y$ if and only if $y \in Tx$.

For topological spaces $X$ and $Y$, a map $T: X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : x \in X, y \in Tx\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in $Y$.

A subset $B$ of a t.v.s. $E$ is *convexly totally bounded* (c.t.b) if for every neighborhood $V$ of the origin 0 of $E$, there exist a finite subset $\{x_i : i \in I\}$ of $E$ and a finite family of convex sets $\{C_i : i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \bigcup\{x_i + C_i : i \in I\}$. See [1].

Let $\mathcal{V}$ be the fundamental system of neighborhoods of the origin 0 in $E$. We recall that a set $K \subset E$ is *locally convex* if for every $x \in K$ and every $V \in \mathcal{V}$, there exists $U \in \mathcal{V}$ such that $\mathrm{co}((x + U) \cap K) \subset x + V$. We say that $K \subset E$ is of *Z type* if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{V}$ such that $\mathrm{co}(U \cap (K - K)) \subset V$. See [4].

The following are noted by Idzik [1]:

1. Every compact set in a locally convex t.v.s is c.t.b.
2. In a locally convex t.v.s., every subset is of $Z$ type and is locally convex.
3. If $K$ is a compact subset of a t.v.s. such that either $K$ is locally convex or of $Z$ type, then it is c.t.b.
4. Every compact convex subset of a t.v.s. $E$ on which $E^*$ separates points is c.t.b. See [5].

The following is a particular form of Idzik [1, Theorem 4.3].

† Supported in part by Ministry of Education, 1996, Project Number BSRI-96-1413.

THEOREM 0. Let $X$ be a nonempty convex subset of a t.v.s. $E$ and $T: X \multimap X$ a closed map with convex values. If $\overline{T(X)}$ is a compact and c.t.b. subset of $X$, then $T$ has a fixed point $x_0 \in X$; that is, $x_0 \in Tx_0$.

In view of the above remarks 1–4, Theorem 0 contains a lot of known results.

## 2. VARIATIONAL INEQUALITIES

From Theorem 0 we deduce a fixed point theorem which is a basis of our variational inequalities.

THEOREM 1. Let $X$ be a convex subset of a t.v.s. $E$, $K$ a nonempty compact subset of $X$, and $T: X \multimap K$ a map such that

(1.1) for each $x \in X$, $Tx$ is convex; and

(1.2) $\{\text{Int}\, T^{-}y\}_{y \in K}$ covers $X$.

If $\overline{T(X)}$ is a c.t.b. subset of $X$, then $T$ has a fixed point.

*Proof.* Note that $\text{co}\, K$ is $\sigma$-compact (see [6]) and hence Lindelöf. Since $E$ is regular, $\text{co}\, K$ is paracompact. Therefore, $T|_{\text{co}\, K}$ has a continuous selection $s$: $\text{co}\, K \to \overline{T(X)} \subset K \subset \text{co}\, K$ [7, Théorème 2.1]. Therefore, by Theorem 0, $s$ has a fixed point. This completes our proof. ■

Theorem 1 is due to Ben-El-Mechaiekh *et al.* [7, Théorème 3.2] whenever $E$ is locally convex. Ben-El-Mechaiekh [8] raised a question whether the local convexity can be eliminated. See [9]. Recently, Kim and Tan [2], Zhang [10], Chang and Zhang [11] used Theorem 1 for a locally convex t.v.s. to obtain a type of variational inequalities.

From Theorem 1 we deduce the following abstract variational inequality.

THEOREM 2. Let $X$ be a convex subset of a t.v.s., $p, q: X \times X \to \mathbf{R}$ functions, and $K$ a nonempty compact c.t.b. subset of $X$. Suppose that

(2.1) $p(x, y) \leq q(x, y)$ for all $(x, y) \in X \times X$ and $q(x, x) \leq 0$ for all $x \in K$;

(2.2) for each $y \in K$, $\{x \in X : p(x, y) > 0\}$ is open in $X$;

(2.3) for each $x \in X$, $\{y \in K : q(x, y) > 0\}$ is convex; and

(2.4) $p(x, y) \leq 0$ for all $x \in X$ and $y \in X \backslash K$.

Then there exists an $x_0 \in X$ such that

$$p(x_0, y) \leq 0 \qquad \text{for all } y \in X.$$

*Proof.* Suppose that for each $x \in X$, there exists a $y \in K$ such that $p(x, y) > 0$ and hence $q(x, y) > 0$ by (2.1). Define $S, T: X \multimap K$ by

$$Sx = \{y \in K : p(x, y) > 0\},$$

$$Tx = \{y \in K : q(x, y) > 0\}$$

for each $x \in X$. Then

(1) for each $x \in X$, $Tx$ is convex by (2.3); and

(2) for each $x \in X$, there exists a $y \in K$ such that $y \in Sx$ or $x \in S^{-}y = \{x \in X : p(x, y) > 0\}$. Since $S^{-}y$ is open in $X$ by (2.2), we have

$$x \in S^{-}y = \text{Int}_X\{x \in X : p(x, y) > 0\} \subset \text{Int}_X T^{-}y.$$

Therefore, by Theorem 1, $T$ has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T\hat{x}$. Hence, $\hat{x} \in K$ and $q(\hat{x}, \hat{x}) > 0$. This contradicts (2.1). Therefore, there exists an $x_0 \in X$ such that

$$p(x_0, y) \leq 0 \qquad \text{for all } y \in K.$$

However, this inequality holds for all $y \in X$ because of (2.4). This completes our proof. ■

Let $\Phi$ be the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

In order to obtain variational inequalities related to multimaps, we need the following simple consequence of the well-known classical result of Berge [12].

LEMMA 3. Let $E$ be a t.v.s. over $\Phi$, $X$ a nonempty subset of $E$, $F$ a topological space, $T: X \multimap F$ an u.s.c. map with compact values, and $\langle\, , \rangle: F \times E \to \Phi$ a function such that for each $y \in E$, $(f, x) \mapsto \mathrm{Re}\langle f, x - y\rangle$ is l.s.c. on $F \times X$. Then for each $y \in E$, the function

$$x \mapsto \inf_{f \in Tx} \mathrm{Re}\langle f, x - y\rangle$$

is l.s.c. on $X$.

Lemma 3 contains Shih and Tan [13, Lemma 2], Ding and Tan [14, Lemma 1], Kim and Tan [2, Lemmas 2 and 4], and Chang and Zhang [11, Lemma 3] as particular cases.

From Lemma 3 we deduce the following.

LEMMA 4. Let $E$ be a t.v.s. over $\Phi$, $F$ a vector space over $\Phi$, and $\langle\, , \rangle: F \times E \to \Phi$ a bilinear function. Let $X$ be a nonempty bounded subset of $E$ such that, for each $z \in F$, $y \mapsto \langle z, y\rangle$ is continuous on $X$. Suppose that $F$ has the $\eta(F, E)$-topology; that is, the topology of uniform convergence on bounded subsets of $E$, and $T: X \multimap F$ is u.s.c. with compact values. Then for each $y \in E$, the function

$$x \mapsto \inf_{f \in Tx} \mathrm{Re}\langle f, x - y\rangle$$

is l.s.c. on $X$.

*Proof.* Similarly as in Kum's paper [15, Lemma B], the pairing $\langle\, , \rangle: F \times X \to \Phi$ is continuous. Therefore, by Lemma 3, we have the conclusion. ■

Note that if $F = E^*$, the topological dual of $E$, then $y \mapsto \langle z, y\rangle$ is obviously continuous for each $z \in E^*$.

Particular forms of Lemma 4 have appeared in Browder [16, Lemma 1], Shih and Tan [17, Lemma 1], Kim and Tan [2, Lemma 2], and Chang and Zhang [11, Lemma 2] with proofs more lengthy than ours.

The following is the main result of this paper.

THEOREM 5. Let $X$ be a bounded convex subset of a t.v.s. $E$ over $\Phi$, $K$ a nonempty compact c.t.b. subset of $X$, $F$ a vector space over $\Phi$, $\langle\, , \rangle: F \times E \to \Phi$ a bilinear function such that $F$ has the $\eta(F, E)$-topology and, for each $z \in F$, $y \mapsto \langle z, y\rangle$ is continuous on $X$, $T: X \multimap F$ be an u.s.c. map with compact values, and $\alpha: X \times X \to \mathbf{R}$ a function such that

(5.1) for each $x \in X$, $\alpha(x, x) = 0$, $\alpha(x, \cdot)$ is concave, and $\alpha(\cdot, x)$ is l.s.c.; and

(5.2) for each $x \in X$ and $y \in X \backslash K$,

$$\inf_{f \in Tx} \operatorname{Re}\langle f, x - y\rangle + \alpha(x, y) \leq 0.$$

Then there exists an $x_0 \in X$ such that

$$\inf_{f \in Tx_0} \operatorname{Re}\langle f, x_0 - y\rangle + \alpha(x_0, y) \leq 0 \qquad \text{for all } y \in X.$$

Moreover, the set of all solutions $x_0$ is a closed subset of $X$. And if $Tx_0$ is convex, then there exists an $f_0 \in Tx_0$ such that

$$\operatorname{Re}\langle f_0, x_0 - y\rangle + \alpha(x_0, y) \leq 0 \qquad \text{for all } y \in X.$$

*Proof.* We use Theorem 2 with $p = q$. Let

$$p(x, y) = \inf_{f \in Tx} \operatorname{Re}\langle f, x - y\rangle + \alpha(x, y).$$

Then we have the following:

(1) $p(x, x) = 0$ by the property of $\langle\, ,\, \rangle$ and (5.1).

(2) For each $y \in K$, $\{x \in X : p(x, y) > 0\}$ is open in $X$ since $x \mapsto p(x, y)$ is l.s.c. on $X$ by Lemma 4 and (5.1).

(3) For each $x \in X$, $\{y \in K : p(x, y) > 0\}$ is convex in $K$. In fact, for any $y_1, y_2 \in K$ satisfying $p(x, y_1) > 0$ and $p(x, y_2) > 0$, let $y = ty_1 + (1 - t)y_2$ for some $t \in (0, 1)$. Then

$$\begin{aligned} p(x, y) &= \inf_{f \in Tx} \operatorname{Re}\langle f, x - (ty_1 + (1 - t)y_2)\rangle + \alpha(x, ty_1 + (1 - t)y_2) \\ &\geq t \inf_{f \in Tx} \operatorname{Re}\langle f, x - y_1\rangle + (1 - t) \inf_{f \in Tx} \operatorname{Re}\langle f, x - y_2\rangle \\ &\quad + t\alpha(x, y_1) + (1 - t)\alpha(x, y_2) \\ &= tp(x, y_1) + (1 - t)p(x, y_2) > 0. \end{aligned}$$

Note that $y \in K$ by (5.2).

(4) For each $y \in X \backslash K$, $p(x, y) \leq 0$ for all $x \in X$ by (5.2).

Therefore by Theorem 2, there exists an $x_0 \in X$ such that

$$p(x_0, y) \leq 0 \qquad \text{for all } y \in X.$$

Moreover, the set of all solutions $x_0$ is

$$\bigcap_{y \in X} \{x \in X : p(x, y) \leq 0\},$$

which is the intersection of nonempty closed sets by (2).

To prove the final assertion, suppose that $Tx_0$ is convex and $\alpha(x_0, \cdot)$ is linear on $X$. We define a function $g: Tx_0 \times X \to \mathbf{R}$ by

$$g(f, y) = \operatorname{Re}\langle f, x_0 - y\rangle + \alpha(x_0, y)$$

for $(f, y) \in Tx_0 \times X$. Then $g$ is linear in $f \in Tx_0$ and in $y \in X$. Note that for a given $y \in X$, $f \mapsto g(f, y)$ is continuous on $F$ with the $\eta(F, E)$-topology. Therefore, by the Kneser minimax theorem [18] or [19, Theorem 4.2], we have

$$\inf_{f \in Tx_0} \sup_{y \in X} g(f, y) = \sup_{y \in X} \min_{f \in Tx_0} g(f, y).$$

Since the right-hand side is $\leq 0$ by the first conclusion, we have

$$\inf_{f \in Tx_0} \sup_{y \in X} g(f, y) \leq 0.$$

Since $f \mapsto \sup_{y \in X} g(f, y)$ is l.s.c. and $Tx_0$ is compact, there exists an $f_0 \in Tx_0$ such that $\sup_{y \in X} g(f_0, y) \leq 0$. This shows the final conclusion. ■

In the case $E$ is locally convex, $F = E^*$, and $\alpha = 0$, Theorem 5 reduces to a recent variational inequality due to Kim and Tan [2, Theorem 1], which extends Browder [20, Theorem 6]. Note that in some cases, we can choose a topology on $F$ different from $\eta(F, E)$ and the assumption on the boundedness of $X$ can be removed from Theorem 2. For example, if we choose the topology $\sigma(F, E)$ (see [21]) or in the case of a normed vector space $E$ and $F = E^*$ (see [2, Theorem 3]), we need not assume the boundedness of $X$.

For a subset $X$ of a t.v.s. $E$, the *inward* set $I_X(x)$ of $X$ at $x \in E$ is defined by

$$I_X(x) = \{x + r(y - x) : r > 0, y \in X\},$$

and $\bar{I}_X(x)$ denotes its closure.

For a locally convex t.v.s. $E$, $F = E^*$, and $\alpha = 0$, we have the following.

COROLLARY 6. Let $X$ be a bounded convex subset of a locally convex t.v.s. $E$, $K$ a nonempty compact subset of $X$, and $T: X \to E^*$ a continuous map, where $E^*$ has the $\eta(E^*, E)$-topology. Suppose that for each $x \in X$ and $y \in X \backslash K$, we have

$$\operatorname{Re}\langle Tx, x - y\rangle \leq 0.$$

Then there exists an $x_0 \in X$ such that

$$\operatorname{Re}\langle Tx_0, x_0 - y\rangle \leq 0 \qquad \text{for all } y \in \bar{I}_X(x_0).$$

Moreover, the set of all solutions $x_0$ is a closed subset of $X$.

*Proof.* In view of Theorem 5, it suffices to show that the inequality of the first part of the conclusion holds for all $y \in I_X(x_0) \backslash X$ since $y \mapsto \langle Tx_0, x_0 - y\rangle$ is continuous. For such $y$, there exist $z \in X$ and $r > 1$ such that $y = x_0 + r(z - x_0)$. Since $r > 1$ and

$$\frac{1}{r} y + \left(1 - \frac{1}{r}\right) x_0 = z \in X,$$

we have

$$0 \geq \operatorname{Re}\langle Tx_0, x_0 - z\rangle = \operatorname{Re}\left\langle Tx_0, \frac{1}{r}(x_0 - y)\right\rangle = \frac{1}{r} \operatorname{Re}\langle Tx_0, x_0 - y\rangle,$$

hence $\operatorname{Re}\langle Tx_0, x_0 - y\rangle \leq 0$. This completes our proof. ■

Note that Corollary 6 strengthens [2, Corollary 2].

## 3. FIXED POINT THEOREMS

In [2], from their own version of Corollary 6, Kim and Tan obtained fixed point theorems for Kakutani maps in a locally convex t.v.s. In this section, we obtain a far-reaching generalization of their theorems.

For any $f \in E^*$ and $U, V \subset E$, we define

$$d_f(U, V) = \inf\{|\mathrm{Re}\langle f, u - v\rangle| : u \in U, v \in V\}.$$

THEOREM 7. Let $X$ be a convex subset of a locally convex t.v.s. $E$, $K$ a nonempty compact subset of $X$, and $R: X \multimap E$ be a map with closed convex values. Suppose that for each $f \in E^*$,

(7.1) $\{x \in X : \inf_{z \in Rx} \mathrm{Re}\langle f, x - z\rangle > 0\}$ is open in $X$;

(7.2) for each $x \in X$, $d_f(Rx, \bar{I}_X(x)) = 0$; and

(7.3) for each $x \in X$ and $y \in X \setminus K$,

$$\inf_{z \in Rx} \mathrm{Re}\langle f, x - z\rangle > 0 \text{ implies } \mathrm{Re}\langle f, x - y\rangle \leq 0.$$

Then $R$ has a fixed point.

*Proof*. Suppose that $R$ has no fixed point. Then by the standard separation theorem on t.v.s., for each $x \in X$, there exists an $f = f_x \in E^*$ such that

$$x \in U_f = \left\{x \in X : \inf_{z \in Rx} \mathrm{Re}\langle f, x - z\rangle > 0\right\}.$$

Note that $D = \mathrm{co}\, K$ is $\sigma$-compact and hence paracompact, as in Theorem 1. By (7.1), $\{U_f : f \in E^*\}$ is an open cover of $D$. Let $\{V_f : f \in E^*\}$ be a locally finite open refinement of the cover $\{U_f : f \in E^*\}$ of $D$ such that $x \in V_f \subset U_f$ where $f = f_x$ for each $x \in D$, and $\{\beta_f : f \in E^*\}$ be the continuous partition of unity subordinated to $\{V_f : f \in E^*\}$. Now define a function $T: D \to E^*$ by

$$Tx = \sum_{f \in E^*} \beta_f(x) f \qquad \text{for } x \in D.$$

For any $x \in D$, we have $\beta_f(x) > 0$ for some $f \in E^*$. Then $x \in V_f \subset U_f$ so that $\inf_{z \in Rx} \mathrm{Re}\langle f, x - z\rangle > 0$. Therefore,

$$\begin{aligned}\inf_{z \in Rx} \mathrm{Re}\langle Tx, x - z\rangle &= \inf_{z \in Rx} \sum_{f \in E^*} \beta_f(x) \mathrm{Re}\langle f, x - z\rangle \\ &\geq \sum_{f \in E^*} \beta_f(x) \inf_{z \in Rx} \mathrm{Re}\langle f, x - z\rangle > 0. \qquad (1)\end{aligned}$$

We now apply Corollary 6. Note that $D$ is bounded, It is standard that $T: D \to E^*$ is continuous, where $E^*$ is equipped with the $\eta(E^*, E)$-topology. Moreover, for each $x \in D$ and $y \in X \setminus K$, we have

$$\mathrm{Re}\langle Tx, x - y\rangle \leq 0. \qquad (2)$$

Otherwise, suppose that

$$\mathrm{Re}\langle Tx, x - y\rangle = \sum_{f \in E^*} \beta_f(x)\, \mathrm{Re}\langle f, x - y\rangle > 0 \tag{3}$$

for some $x \in D$ and $y \in X\backslash K$. If $\beta_f(x) > 0$, then $\inf_{z \in Rx} \mathrm{Re}\langle f, x - z\rangle > 0$. Hence, by (7.3), we have $\mathrm{Re}\langle f, x - y\rangle \le 0$, which contradicts (3).

Therefore, by Corollary 6, there exists an $\hat{x} \in D$ such that

$$\mathrm{Re}\langle T\hat{x}, \hat{x} - y\rangle \le 0 \qquad \text{for all } y \in D. \tag{4}$$

In view of (2), this inequality also holds for $y \in X\backslash D \subset X\backslash K$, hence for all $y \in \bar{I}_X(\hat{x})$ as in the proof of Corollary 6.

Let $\delta = \inf_{z \in R(\hat{x})} \mathrm{Re}\langle T\hat{x}, \hat{x} - z\rangle > 0$ in (1). For any $\varepsilon > 0$ with $\delta > \varepsilon$, by (7.2), there exist a $\hat{z} \in R\hat{x}$ and a $y_\varepsilon \in \bar{I}_X(\hat{x})$ such that

$$|\mathrm{Re}\langle T\hat{x}, \hat{z} - y_\varepsilon\rangle| \le \varepsilon.$$

Then

$$\begin{aligned}\mathrm{Re}\langle T\hat{x}, \hat{x} - y_\varepsilon\rangle &= \mathrm{Re}\langle T\hat{x}, \hat{x} - \hat{z}\rangle + \mathrm{Re}\langle T\hat{x}, \hat{z} - y_\varepsilon\rangle \\ &\ge \delta - \varepsilon > 0,\end{aligned}$$

which contradicts (4). This completes our proof. ■

Recall that a map $R: X \multimap E$ satisfying (7.1) for each $f \in E^*$ is said to be *generalized upper hemicontinuous*. This class of maps properly contains upper hemicontinuous (u.h.c.) maps. The fixed point theory on generalized u.h.c. maps was extensively studied in [22–25].

Note that Kim and Tan [2, Theorem 2] obtained Theorem 7 under the assumption that $X$ is paracompact and bounded under other restrictions. For a normed vector space $E$, Theorem 7 reduces to [2, Theorem 4], where $R$ was assumed to be u.h.c.

Moreover, condition (7.2) is equivalent to

(7.2)′ for each $x \in \mathrm{Bd}\, X$, $d_f(Rx, \bar{I}_X(x)) = 0$,

since $I_X(x) = E$ for $x \in \mathrm{Int}\, X$.

If $X = K$ in Theorem 7, we have another result as follows.

THEOREM 8. Let $X$ be a nonempty compact convex subset of a t.v.s. $E$ on which $E^*$ separates points, and $R: X \multimap E$ be a map with compact convex values. Suppose that, for each $f \in E^*$,

(8.1) $\{x \in X : \inf_{z \in Rx} \mathrm{Re}\langle f, x - z\rangle > 0\}$ is open in $X$; and

(8.2) for each $x \in X$, $d_f(Rx, \bar{I}_X(x)) = 0$.

Then $R$ has a fixed point.

*Proof.* Follow the proof of Theorem 7 and use Theorem 5 with $X = K$, $\alpha = 0$ and $F = E^*$, instead of Corollary 6. ■

Note that Theorem 8 was due to Park [22–24]. Theorem 7 with $X = K$ and Theorem 8 generalize the historically well-known fixed point theorems due to Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Browder, and Halpern for u.s.c. maps, to Fan, Reich, Granas and Liu for upper demicontinuous (u.d.c.) maps, to Cornet, Lasry and Robert for u.h.c. maps, and to Glebov, Cellina, Simons, and Park for generalized u.h.c. maps. Those results extend the corresponding single-valued versions due to many authors including Brouwer, Schauder, and Tychonoff. For details, see [22, 24].

## 4. GENERALIZED VARIATIONAL INEQUALITIES

Recently, Ding [3] studied a class of generalized nonlinear variational inequality problems (GNVIP) as follows:

To find $\hat{y} \in X$ and $\hat{w} \in T\hat{y}$ such that

$$\operatorname{Re}\langle \hat{w}, x - g\hat{y}\rangle + b(\hat{y}, x) - b(\hat{y}, \hat{y}) \geq 0 \qquad \text{for all } x \in X,$$

where $X$ is a nonempty convex subset of a t.v.s. $E$, $T: X \multimap E^*$, $g: X \to E$ and $b: X \times X \to \mathbf{R}$ are maps.

Actually, Ding [3] extended all of the results in [2] by considering a continuous map $g: X \to E$ with bounded range. If we choose $g = 1_X$ and $b = 0$, all of Ding's results reduce to the corresponding ones in [2].

In this section, we indicate that the key results of Ding [3] can be extended and improved adopting our method.

LEMMA 4′. Under the hypothesis of Lemma 4, further assume that $g: X \to E$ is a continuous map with bounded range. Then for each $y \in E$, the function

$$X \mapsto \inf_{f \in Tx} \operatorname{Re}\langle f, gx - y\rangle$$

is l.s.c. on $X$.

Lemma 4′ reduces to Lemma 4 if $g = 1_X$ and to Ding [3, Lemma 1.2] if $F = E^*$.

The following is a solution of the GNVIP.

THERoREM 5′. Under the hypothesis of Theorem 5, further assume that $g: X \to E$ is a continuous map with bounded range and the following instead of (5.2):

(5′.2) for each $x \in X$ and $y \in X \backslash K$,

$$\inf_{f \in Tx} \operatorname{Re}\langle f, gx - y\rangle + \alpha(x, y) \leq 0.$$

Furthermore, assume that

(5′.4) for each $x \in X$, there exists an $f \in Tx$ such that $\operatorname{Re}\langle f, gx\rangle \leq \operatorname{Re}\langle f, x\rangle$. Then there exists an $x_0 \in X$ such that

$$\inf_{f \in Tx_0} \operatorname{Re}\langle f, gx_0 - y\rangle + \alpha(x_0, y) \leq 0 \qquad \text{for all } y \in X.$$

Moreover, the set of all solutions $x_0$ is a closed subset of $X$. And if $Tx_0$ is convex, then there exists an $f_0 \in Tx_0$ such that

$$\operatorname{Re}\langle f_0, gx_0 - y\rangle + \alpha(x_0, y) \leq 0 \qquad \text{for all } y \in X.$$

Theorem 5′ reduces to Theorem 5 if $g = 1_X$ and extends Ding [3, Theorem 2.1] even if $E$ is locally convex and $F = E^*$. In fact, he considered the case

$$\alpha(x, y) = b(x, x) - b(x, y)$$

where $b: X \times X \to \mathbf{R}$ is a continuous function such that $b(x, \cdot)$ is a convex function.

COROLLARY 6′. Let $X$, $E$, $K$ and $T$ be the same as in Corollary 6, and $g: X \to E$ a continuous map with bounded range. Suppose that

(6′.1) for each $x \in X$, we have $\mathrm{Re}\langle Tx, gx\rangle \leq \mathrm{Re}\langle Tx, x\rangle$; and

(6′.2) for each $x \in X$ and $y \in X \backslash K$, we have $\mathrm{Re}\langle Tx, gx - y\rangle \leq 0$.

Then there exists an $x_0 \in X$ such that

$$\mathrm{Re}\langle Tx_0, gx_0 - y\rangle \leq 0 \qquad \text{for all } y \in \bar{I}_X(gx_0).$$

Moreover, the set of all solutions $x_0$ is a closed subset of $X$.

If $g = 1_X$, then Corollary 6′ reduces to Corollary 6.

The following is a coincidence theorem.

THEOREM 7′. Let $X$, $E$, $K$ and $R$ be the same as in Theorem 7, and $g: X \to E$ a continuous map with bounded range. Suppose that for each $f \in E^*$,

(7′.1) $\{x \in X : \inf_{z \in Rx} \mathrm{Re}\langle f, gx - z\rangle < 0\}$ is open in $X$;

(7′.2) for each $x \in X$, $d_f(Rx, \bar{I}_X(gx)) = 0$;

(7′.3) for each $x \in X$ and $y \in X \backslash K$,

$$\inf_{z \in Rx} \mathrm{Re}\langle f, gx - z\rangle > 0 \quad \text{implies} \quad \mathrm{Re}\langle f, gx - y\rangle \leq 0;$$

(7′.4) for each $x \in K$,

$$\inf_{z \in Rx} \mathrm{Re}\langle f, gx - z\rangle > 0 \quad \text{implies} \quad \mathrm{Re}\langle f, gz - x\rangle \leq 0.$$

Then there exists an $\hat{x} \in X$ such that $g\hat{x} \in R\hat{x}$.

Note that if $R$ is u.h.c., then (7′.1) holds. Theorem 7′ reduces to Theorem 7 if $g = 1_X$ and extends Ding [3, Theorem 3.2], where $X$ is assumed to be paracompact and bounded under some restricted form of (7′.2).

Other results in [3] are consequences of Theorems 5′ and 7′.

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