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THE SCHAUDER TYPE AND OTHER FIXED POINT THEOREMS IN HYPERCONVEX SPACES*

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Abstract. For hyperconvex metric spaces, we obtain generalized KKM theorems, the matching theorem for open covers, the Fan-Browder type coincidence theorems, the Schauder type and other fixed point theorems.

1. Introduction

The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi [1] in 1956. In 1979, independently Sine [21] and Soardi [24] proved the fixed point property for nonexpansive maps on bounded hyperconvex spaces. Since then many interesting works have appeared on the study of hyperconvex spaces concentrated to mainly the relationship with nonexpansive maps. For example, see [2, 3, 8-15, 22, 23].

Recently, Khamsi [11] established the Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem) for hyperconvex spaces and applied it to prove an analogue of Ky Fan's best approximation theorem extending the Brouwer and the Schauder fixed point theorems. Motivated by [11], the present author [18] applied Khamsi's KKM theorem

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to obtain a Ky Fan type matching theorem for open covers, a coincidence theorem, a Fan-Browder type fixed point theorem, a Brouwer-Schauder-Rothe type fixed point theorem, and other results on mainly compact hyperconvex spaces.

On the other hand, Horvath [4-8] initiated study of his C-spaces, which are meaningful generalizations of convex spaces or convex subsets of topological vector spaces. Moreover, in [8], he found that hyperconvex spaces are particular type of C-spaces. Recently, C-spaces are generalized to G-convex spaces by the present author [19, 20] and foundations of the KKM theory were established for convex spaces in [17] and for G-convex spaces in [20]. Therefore, we can apply the results in [20] to hyperconvex spaces.

In the present paper, we continue the study of hyperconvex spaces in [18] and obtain noncompact versions of the KKM theorem, the matching theorem for open covers, the Fan-Browder type coincidence theorems and other results. Especially, we obtain the Schauder type fixed point theorems for compact maps on hyperconvex spaces and a generalization of a theorem due to Kirk [14] on the location of fixed point sets.

Our arguments are based on the KKM theorem due to Khamsi [11] and a lemma due to Espinola-Garcia [3] on the invariance of the measure of noncompactness under passage to the hyperconvex hull.

2. Preliminaries

A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

For any nonempty bounded subset A of H, its *convex hull* $\operatorname{co} A$ is defined by

$$\operatorname{co} A = \bigcap \{B : B \text{ is a closed ball containing } A\}.$$

Let $\mathcal{A}(H) = \{A \subset H : A = \operatorname{co} A\}$; that is, $A \in \mathcal{A}(H)$ iff A is an intersection of balls. In this case we will say A is an *admissible* subset of H; see [11].

For any $X \subset H$, a multimap (or a map) $G : X \multimap H$ is called a *KKM-map* if

$$\operatorname{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$$

for any $x_1, \ldots, x_n \in X$.

The following KKM-theorem is due to Khamsi [11, Theorem 4]:

THEOREM 0. *Let H be a hyperconvex space and $X \subset H$ a subset. Let $G : X \multimap H$ be a KKM map such that $G(x)$ is closed for any $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then we have*

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

For a map $G : X \multimap Y$, we denote $x \in G^{-1}(y)$ iff $y \in G(x)$ where $x \in X$ and $y \in Y$. Let $\mathbb{C}(X, Y)$ denote the class of single-valued continuous maps $f : X \to Y$.

Let μ be either the Kuratowski (set) or Hausdorff (ball) measure on the family of bounded subset of a metric space H.

The following is known [3, Lemma 5 and Corollary]:

LEMMA 1. *Let H be a hyperconvex space and Y a subset of H. Then there exists $Z \subset H$ such that $Y \subset Z \subset H$, Z is isometric to εY, and $\mu(Y) = \mu(Z)$.*

Here, εY denotes the hyperconvex hull of Y introduced by Isbell [9]; see also [3]. Note that Z is a hyperconvex subset of H.

Let $\langle X \rangle$ denote the class of nonempty finite subsets of a set X.

In view of Lemma 1 or [9, Proposition 2.11], for any $N \in \langle H \rangle$ and a compact subset L of a hyperconvex space H, there is a compact hyperconvex subset L_N of H isometric to $\varepsilon(L \cup N)$.

3. General KKM theorems

In this section, we deduce useful generalized forms of the KKM type theorems.

From Theorem 0, we have the following:

Theorem 1. *Let H be a hyperconvex space, $X \subset H$, and $G : X \multimap H$ a KKM map with compactly closed values. Then for every compact hyperconvex subset $H_0 \subset H$, we have*

$$H_0 \cap \bigcap\{G(x) : x \in H_0 \cap X\} \neq \emptyset.$$

Proof. Define $G_0(x) = G(x) \cap H_0$ for $x \in H_0 \cap X$. Then $G_0 : H_0 \cap X \multimap H_0$ is well-defined. Consider $(H_0, H_0 \cap X, G_0)$ instead of (H, X, G) in Theorem 0. Then all of the requirements are satisfied. Therefore, we have

$$\bigcap\{G_0(x) : x \in H_0 \cap X\} = H_0 \cap \{G(x) : x \in H_0 \cap X\} \neq \emptyset.$$

This completes our proof.

From Theorem 1, we have the following:

Theorem 2. *Let H be a hyperconvex space, $X \subset H$, Y a topological space, $t \in \mathbb{C}(H, Y)$, $G : X \multimap Y$ a map, and K a nonempty compact subset of Y. Suppose that*

(2.1) *for each $x \in X$, $G(x)$ is compactly closed;*

(2.2) *$t^{-1}G : X \multimap H$ is a KKM map; and*

(2.3) *for any $N \in \langle X \rangle$, there exists a compact hyperconvex subset $L_N \subset H$ containing N such that $t(L_N) \cap \bigcap\{G(x) : x \in L_N \cap X\} \subset K$.*

Then we have

$$\overline{t(H)} \cap K \cap \bigcap\{G(x) : x \in X\} \neq \emptyset.$$

Proof. Suppose that the conclusion does not hold. Since $\overline{t(H)} \cap K$ is compact and contained in $\bigcup\{Y \backslash G(x) : x \in X\}$, by (2.1), there exists an $N \in \langle X \rangle$ such that

$$\overline{t(H)} \cap K \subset \bigcup_{x \in N} (Y \backslash G(x)).$$

For the $L_N \subset H$ in (2.3), this implies

$$L_N \cap \bigcap_{x \in L_N \cap X} t^{-1}G(x) \cap t^{-1}(K) = \emptyset.$$

However, by (2.3), we have

$$L_N \cap \bigcap_{x \in L_N \cap X} t^{-1}G(x) \subset t^{-1}(K).$$

Therefore, we have

$$L_N \cap \bigcap_{x \in L_N \cap X} t^{-1}G(x) = \emptyset,$$

which contradicts Theorem 1. This completes our proof.

Remark. For $H = Y$, $t = 1_H$ the identity map of H, and $K = G(x_0)$ for some $x_0 \in X$, Theorem 2 reduces to Theorem 0. Therefore, in Theorem 0, G may have compactly closed values.

4. Matching and coincidence theorems

Theorem 2 can be restated in its contrapositive form and in terms of the complement Sx of Gx in Y as follows:

Theorem 3. *Let H be a hyperconvex space, $X \subset H$, Y a topological space, $t \in \mathbb{C}(H, Y)$, $S : X \multimap Y$ a map, and K a nonempty compact subset of Y. Suppose that*

(3.1) *for each $x \in X$, Sx is compactly open;*
(3.2) *$\overline{t(H)} \cap K \subset S(X)$; and*
(3.3) *for each $N \in \langle X \rangle$, there exists a compact hyperconvex subset $L_N \subset H$ containing N such that $t(L_N) \backslash K \subset S(L_N \cap X)$.*

Then there exist an $M \in \langle X \rangle$ and an $x_0 \in \operatorname{co} M$ such that $t(x_0) \in \bigcap_{x \in M} S(x)$; that is, $t(\operatorname{co} M) \cap \bigcap_{x \in M} S(x) \neq \emptyset$.

Remark. Theorem 3 is a particular form of Ky Fan's matching theorem for open covers in the KKM theory of generalized convex spaces; see [16, 17, 19, 20].

For $K = Y$, Theorem 3 reduces to [18, Theorem 1].

From Theorem 3, we have the following Fan-Browder type coincidence theorem:

Theorem 4. *Let H, X, Y, t, and K be the same as in Theorem 3. Suppose that two multimaps $S : X \multimap Y$ and $T : H \multimap Y$ satisfy the following:*

(4.1) *for each $x \in X$, $S(x) \subset T(x)$ and $S(x)$ is compactly open;*
(4.2) *for each $y \in t(H)$, $T^{-1}(y)$ is admissible;*
(4.3) *$\overline{t(H)} \cap K \subset S(X)$; and*
(4.4) *for each $N \in \langle X \rangle$, we have an L_N as in* (3.3) *such that $t(L_N) \backslash K \subset S(L_N \cap X)$.*

Then there exists an $x_0 \in H$ such that $t(x_0) \in T(x_0)$.

Proof. Since (4.1), (4.3), and (4.4) imply (3.1)-(3.3), by Theorem 3, there exist an $M \in \langle X \rangle$ and an $x_0 \in \operatorname{co} M$ such that $t(x_0) \in \bigcap_{x \in M} S(x)$. Therefore, $t(x_0) \in \bigcap_{x \in M} T(x)$ by (4.1). Since $y = t(x_0) \in t(H)$ and $x \in T^{-1}(y)$ for all $x \in M$, by (4.2), we have $\operatorname{co} M \subset T^{-1}(y)$. In particular, $x_0 \in T^{-1}(y)$; that is, $y \in T(x_0)$. This completes our proof.

Remark. For $K = Y$, Theorem 4 reduces to [18, Theorem 2] and, for $H = X = Y = K$, $t = 1_H$, and $S(x) = \operatorname{Int} T(x)$, Theorem 4 reduces to [18, Theorem 3].

So far, from Theorem 0, we deduced Theorems 1-4, which are noncompact versions of corresponding results in [6, 13]. For far-reaching generalized forms of Theorems 0-4 for G-convex spaces, see [15].

5. Fixed points of nonself maps

As another application of Theorem 3, we obtain the following Fan type best approximation theorem for hyperconvex spaces:

THEOREM 5. *Let H be a hyperconvex space, $X \in \mathcal{A}(H)$, and $f \in \mathbb{C}(X, H)$. Suppose that there exists a nonempty compact subset K of X such that*

(0) *for each $N \in \langle X \rangle$, there exists a compact hyperconvex subset $L_N \subset X$ containing N such that for each $x \in L_N \backslash K$, $d(y, f(x)) < d(x, f(x))$ holds for some $y \in L_N$.*

Then either f has a fixed point $x_0 \in K$; or there exists an $x_0 \in K \cap \operatorname{Bd} X$ such that

$$0 < d(x_0, f(x_0)) \le d(y, f(x_0)) \quad \text{for all} \quad y \in X.$$

Proof. Suppose that for each $x \in K$, there exists a $y \in X$ such that

$$d(x, f(x)) > d(y, f(x)).$$

Define $S : X \multimap X$ by

$$S(x) := \{y \in X : d(x, f(y)) < d(y, f(y))\}$$

for $x \in X$. Since f is continuous, $S(x)$ is open in X. Moreover, for each $x \in K$, we have a $y \in X$ such that $x \in S(y)$. Therefore, (3.1) and (3.2) are satisfied. Note that condition (0) satisfies (3.3) with $t = 1_X$. Therefore, by Theorem 3 with $H = X = Y$ and $t = 1_X$, there exist an $M \in \langle X \rangle$ and an $x_* \in \operatorname{co} M$ such that $x_* \in \bigcap_{x \in M} S(x)$; that is, $d(x_i, f(x_*)) < d(x_*, f(x_*))$ for each $x_i \in M := \{x_1, \ldots, x_n\} \in \langle X \rangle$.

Let $\varepsilon > 0$ such that $d(x_i, f(x_*)) \le d(x_*, f(x_*)) - \varepsilon$. Then

$$x_i \in B(f(x_*), d(x_*, f(x_*)) - \varepsilon).$$

Since the closed ball is admissible and contains M, we have

$$x_* \in \operatorname{co} M \subset B(f(x_*), d(x_*, f(x_*)) - \varepsilon).$$

This is a contradiction. Hence there exists an $x_0 \in K$ such that $f(x_0, f(x_0)) \le d(y, f(x_0))$ for all $y \in X$.

Suppose that $x_0 \in \operatorname{Int} X$. Then there exists an $r > 0$ such that

$$B(x_0, r) \subset X \text{ and } r < d(x_0, f(x_0)) \le d(y, f(x_0)) \text{ for all } y \in B(x_0, r).$$

Then there is a $y_0 \in B(x_0, r) \cap B(f(x_0), d(x_0, f(x_0)) - r)$ by the hyperconvexity of H. Hence,

$$d(y_0, f(x_0)) \le d(x_0, f(x_0)) - r < d(x_0, f(x_0)),$$

which is a contradiction. Therefore, $x_0 \in \operatorname{Bd} X$.

This completes our proof.

Remark. For $X = K$, condition (0) holds trivially and Theorem 5 reduces to Khamsi [11, Lemma].

From Theorem 5, we have the following fixed point theorem:

Theorem 6. *Under the hypothesis of Theorem 5, f has a fixed point if one of the following conditions holds for all $x \in K \cap \operatorname{Bd} X$ such that $x \neq f(x)$:*

(i) *There exists a $y \in X$ such that*

$$d(x, f(x)) > d(y, f(x)).$$

(ii) *There exists a $\alpha \in (0, 1)$ such that*

$$X \cap B(f(x), \alpha d(x, f(x)) \neq \emptyset.$$

(iii) $f(x) \in X$.

Proof. (i) Suppose that f has no fixed point in K. Then by Theorem 5, there exists an $x_0 \in K \cap \operatorname{Bd} X$ such that

$$0 < d(x_0, f(x_0)) \leq d(y, f(x_0)) \quad \text{for all} \quad y \in X.$$

This contradicts condition (i).

(ii) For any $x \in K \cap \operatorname{Bd} X$ with $x \neq f(x)$, there exists a $y \in X$ such that

$$y \in X \cap B(f(x), \alpha d(x, f(x)).$$

Then

$$d(y, f(x)) \leq \alpha d(x, f(x)) < d(x, f(x)).$$

Therefore (ii) implies (i).

(iii) Clearly (iii) implies (i).

Remark. For $X = K$, Theorem 6 reduces to [18, Theorem 5] including Khamsi [11, Theorem 6] and Espinola-Garcia [3, Lemma 2]. Note that Theorem 6(iii) can be regarded as a far-reaching generalization of the Brouwer, Schauder, or Rothe type fixed point theorems for hyperconvex spaces.

6. The Schauder type fixed point theorem

As another application of Theorem 4, we have the following fixed point result on compact maps:

Lemma 2. *Let H be a hyperconvex space, $X \subset H$, K a nonempty compact subset of H, and $t \in \mathbb{C}(H, K)$. Suppose that, for each $\delta > 0$, there exist two multimaps $S : X \multimap H$ and $T : H \multimap H$ satisfying (4.1)-(4.3) of Theorem 4 and $d(x, y) < \delta$ for all $x \in H$ and $y \in T(x)$. Then t has a fixed point.*

Proof. Note that $t(H) \subset K$ implies condition (4.4) of Theorem 4. Then, by Theorem 4, for any $\delta > 0$, there exists an $x_0 \in H$ such that $t(x_0) \in T(x_0)$ and $d(x_0, t(x_0)) < \delta$. Therefore, for any $\delta > 0$, t has an δ-fixed point. Since $\overline{t(H)} \subset K$ is compact, t must have a fixed point.

From Lemma 2, we deduce the following Schauder fixed point theorem for compact maps defined on hyperconvex spaces:

Theorem 7. *Let H be a hyperconvex space and $f \in \mathbb{C}(H, H)$. If f is compact, then f has a fixed point.*

Proof. For any $\delta > 0$, define two maps $S, T : H \multimap H$ by

$$S(x) = \{y \in H : d(x, y) < \delta/2\} \text{ and } T(x) = \{y \in H : d(x, y) \leq \delta/2\}$$

for $x \in H$. Let $K = \overline{f(H)}$. Then (4.1) holds clearly. For each $y \in H$, $T^{-}(y) = \{x \in H : d(x, y) \leq \delta/2\} = B(y, \delta/2)$ is admissible. Hence (4.2) holds. Clearly (4.3) and (4.4) are satisfied with $t = f$. Moreover, $d(x, y) < \delta$ for all $x \in H$ and $y \in T(x)$. Therefore, by Lemma 2, f has a fixed point.

Note that Theorem 7 can also be derived from Theorem 6(iii) as follows:

Another Proof. Since $K = \overline{f(H)}$ is compact, by Lemma 1 or Isbell [9, Proposition 2.11], there is a compact hyperconvex subset X of H containing K such that X is isometric to εK. Then, by Theorem 6(iii) with $X = K$, $f|_X \in \mathbb{C}(X, X)$ has a fixed point.

REMARK. For the case H is bounded, Espinola-Garcia [3, Theorem] obtained a generalization of Theorem 7 for a μ-condensing map f.

Finally, we have a result closely related to Theorem 7:

THEOREM 8. *Let H be a hyperconvex space and $f \in \mathbb{C}(H, H)$ compact. Let H' be a compact hyperconvex subset of H containing $\overline{f(H)}$ such taht H' is isometric to $\varepsilon(\overline{f(H)})$. Let $\mathcal{T}$ be the class of nonempty admissible subsets of H' endowed with the Hausdorff metric. Define a map $\overline{f} : \mathcal{T} \to \mathcal{T}$ by setting*

$$\overline{f}(X) = \operatorname{co} f(X) \quad \textit{for} \quad X \in \mathcal{T}.$$

Set $D_0 = H'$, let $D_n = \overline{f}(D_{n-1}) = \overline{f}^n(H')$, and suppose $D = \bigcap_{n=0}^{\infty} D_n$. Then $\overline{f}(D) = D$, and $D = \lim_{n\to\infty} D_n$, where the limit is taken relative to the Hausdorff metric on $\mathcal{T}$. In particular, if $f(x) = x$ then $x \in D$.

REMARK. The existence of H' is a consequence of Lemma 1. Kirk [14, Theorem 1] obtained Theorem 8 for the case H itself is compact, from which Theorem 8 clearly follows.

Note that our Schauder Theorem 7 provides the nonempty fixed point set of a compact map $f \in \mathbb{C}(H, H)$ and Theorem 8 its approximate location.

Kirk [14] also noted that Theorem 8 can be thought of as an abstract formulation of a well known fact in interval analysis.

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