# FIXED POINTS IN HOMEOMORPHICALLY CONVEX SETS

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Abstract. We obtain new fixed point theorems for the admissible class $\mathfrak{A}_c^\kappa$ of multimaps defined on admissible subsets $X$ (in the sense of Klee) of not-necessarily locally convex topological vector spaces. It is shown also that $X$ can be homeomorphically convex.

*Key Words and Phrases.* Multimap (closed, compact, u.s.c., l.s.c., continuous), acyclic, polytope, admissible class of multimaps, admissible set (in the sense of Klee).

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## 1. Introduction and Preliminaries

In this paper, we obtain new fixed point theorems for the admissible class $\mathfrak{A}_c^\kappa$ of multimaps defined on admissible subsets (in the sense of Klee) of not-necessarily locally convex topological vector spaces. Our new results properly generalize a large number of historically well-known theorems.

A *multimap* or *map* $T : X \multimap Y$ is a function from $X$ into the power set of $Y$ with nonempty values, and $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

Given two maps $T : X \multimap Y$ and $S : Y \multimap Z$, their *composition* $ST : X \multimap Z$ is defined by $(ST)(x) = S(T(x))$ for $x \in X$.

For topological spaces $X$ and $Y$, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in $Y$.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of $X$; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^{-1}(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that composites of u.s.c. maps are u.s.c.; the image of a compact set under an u.s.c. map with compact values is compact; and every u.s.c. map $T$ with closed values is closed.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any convex or star-shaped subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \multimap Y$ is said to be *acyclic* if it is u.s.c. with compact acyclic values.

Throughout this paper, t.v.s. means Hausdorff topological vector spaces, and co denotes the convex hull. A *polytope* is a convex hull of a nonempty finite subset of a t.v.s. or a compact convex subset of a finite dimensional subspace.

For any topological spaces $X$ and $Y$ and given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope $P$, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex sets), the Aronszajn maps $\mathbb{M}$ (with $R_\delta$ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values consisting of one or $m$ acyclic components, where $m$ is fixed), the approachable maps $\mathbb{A}$ in t.v.s., admissible maps in the sense of Górniewicz, permissible maps of Dzedzej; for references, see [P1,5].

We introduce two more classes:

$F \in \mathfrak{A}_c^\sigma(X,Y) \iff$ for any $\sigma$-compact subset $K$ of $X$, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X,Y) \iff$ for any compact subset $K$ of $X$, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

Note that $\mathbb{K}_c^\sigma$ due to Lassonde [L] and $\mathbb{V}_c^\sigma$ due to Park *et al.* [PSW] are examples of $\mathfrak{A}_c^\sigma$.

An approximable map defined by Ben-El-Mechaiekh and Idzik [BI] belongs to $\mathfrak{A}_c^\kappa$. Moreover, any u.s.c. compact map defined on a closed subset of a locally convex t.v.s. with closed values is approximable whenever its values are all (1) convex, (2) contractible, (3) decomposable, or (4) $\infty$-proximally connected; see [BI].

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Any class $\mathfrak{A}_c^\kappa$ will be called *admissible.* For details, see [P1-3, PK1,2].

A nonempty subset $X$ of a t.v.s. $E$ is said to be *admissible* (in the sense of Klee [K]) provided that, for every compact subset $K$ of $X$ and every neighborhood $V$ of the origin 0 of $E$, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace $L$ of $E$.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are $l^p$, $L^p$, the Hardy spaces $H^p$ for $0 < p < 1$, and the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$. Moreover, any locally convex subset of an $F$-normable t.v.s. and any compact convex locally convex subset of a t.v.s. are admissible. An example of a nonadmissible nonconvex compact subset of the Hilbert space $l^2$ is known. For details, see Hadžić [H], Weber [W1,2], and references therein.

## 2. Main Results

In our previous works [P1,2], it is shown that if $X$ is a nonempty convex subset of a locally convex t.v.s., then any compact map in $\mathfrak{A}_c^\sigma(X,X)$ has a fixed point, and furthermore if $X$ is compact, then any map in $\mathfrak{A}_c^\kappa(X,X)$ has a fixed point. Those two results are extended as follows:

**Theorem 1.** *Let $E$ be a t.v.s. and $X$ an admissible convex subset of $E$. Then any compact map $T \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

*Proof.* Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin $0$ of $E$ and let $V \in \mathcal{V}$. Since $\overline{T(X)}$ is a compact subset of the admissible subset $X$, there exist a continuous function $f : \overline{T(X)} \to X$ and a finite dimensional subspace $L$ of $E$ such that $x - f(x) \in V$ for all $x \in \overline{T(X)}$ and $f(\overline{T(X)}) \subset L \cap X$. Let $M := f(\overline{T(X)})$. Then $M$ is a compact subset of $L$ and hence $K := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $f : \overline{T(X)} \to K$ and $T|_K : K \to \overline{T(X)}$. Since $T \in \mathfrak{A}_c^\kappa(X,X)$ and $K$ is a compact subset of $X$, there exists a map $\Gamma \in \mathfrak{A}_c(K, \overline{T(X)})$ such that $\Gamma(x) \subset T(x)$ for all $x \in K$. Then the composite $f\Gamma : K \to K$ belongs to $\mathfrak{A}_c(K,K)$ and hence, has a fixed point $x_V \in f\Gamma(x_V)$. Let $x_V = f(y_V)$ for some $y_V \in \Gamma(x_V) \subset \overline{T(X)}$. We have $y_V - f(y_V) = y_V - x_V \in V$. Since $\overline{T(X)}$ is compact, we may assume that $y_V$ converges to some $\hat{x}$. Then $x_V$ also converges to $\hat{x}$ and hence $\hat{x} \in K$. Since the graph of $\Gamma$ is closed in $K \times \Gamma(K)$, we have $\hat{x} \in \Gamma(\hat{x}) \subset T(\hat{x})$. This completes our proof.

*Remark.* As we have seen in our previous works [P1,2], Theorem 1 is a far-reaching generalization of historically well-known results due to Brouwer, Schauder, Tychonoff, Mazur, Kakutani, Hukuhara, Bohnenblust and Karlin, Fan, Glicksberg, Rhee, Himmelberg, Powers, Granas and Liu, Simons, and Lassonde. For the literature, see [P3,5].

A particular form of Theorem 1 for acyclic maps was due to Park [P4], and applied to existence of solutions of quasi-equilibrium problems.

As an application of Theorem 1, we show that the convexity of the set $X$ in Theorem 1 is not essential. In fact, Theorem 1 holds for homeomorphically convex sets as follows:

**Theorem 2.** *Let $E$ and $F$ be t.v.s. and $X$ a subset of $E$ which is homeomorphic to an admissible convex subset $\Delta$ of $F$. Then any compact map $T \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

*Proof.* Let $h : \Delta \to X$ be the homeomorphism. Then the composite $h^{-1}Th : \Delta \to \Delta$ belongs to $\mathfrak{A}_c^\kappa(\Delta, \Delta)$. Since $T$ is compact, so is $h^{-1}Th$. Therefore, by Theorem 1, there exists an $z_0 \in \Delta$ such that $z_0 \in h^{-1}Th(z_0)$ or equivalently $h(z_0) \in Th(z_0)$. Hence $\hat{x} = h(z_0)$ is a fixed point of $T$. This completes our proof.

*Remark.* Theorem 2 is motivated by recent works of Clarke, Ledyaev, and Stern [C1,2] on the existence of zeros and fixed points of multimaps in nonconvex sets.

As an application of Theorem 2, we have the following new Fan-Browder type fixed point theorem for compact maps:

**Theorem 3.** *Let $E$ be a t.v.s. and $X$ a subset of $E$ which is homeomorphic to an admissible convex subset of some t.v.s. Let $S, T : X \multimap X$ be compact maps such that*

(1) *for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and*
(2) *$\{\operatorname{Int} S^{-1}(y)\}_{y \in X}$ covers $X$.*

*Then $T$ has a fixed point.*

*Proof.* It is well-known that, for each compact subset $K$ of $X$, the map $T|_K$ has a continuous selection. Then $T \in \mathbb{C}_c^\kappa(X, X) \subset \mathfrak{A}_c^\kappa(X, X)$. Therefore, by Theorem 2, $T$ has a fixed point.

*Remark.* If $X$ itself is compact and convex, then Theorem 3 holds without assuming the admissibility of $X$. This is usually called the Fan-Browder fixed point theorem and has numerous applications. For far-reaching generalizations of the theorem, see Park and Kim [PK2]. Note that Ben-El-Mechaiekh [B] obtained a particular form of Theorem 3 for a locally convex t.v.s. $E$.

Now, we raise the following general form of the Schauder conjecture:

**Problem.** *Does a convex subset of a (metrizable) t.v.s. have the fixed point property for compact maps in $\mathfrak{A}_c^\kappa$?*

If an answer is affirmative, then admissibility can be eliminated in Theorems 1-3.

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